

# Comments on the String dual to $N = 1$ SQCD

Carlos Hoyos <sup>1</sup>, Carlos Núñez <sup>2</sup> and Ioannis Papadimitriou<sup>3</sup>

*Department of Physics  
University of Swansea, Singleton Park  
Swansea SA2 8PP  
United Kingdom.*

## Abstract

We study the String dual to  $N = 1$  SQCD deformed by a quartic superpotential in the quark superfields. We present a unified view of the previous results in the literature and find new *exact* solutions and new asymptotic solutions. Then we study the Physics encoded in these backgrounds, giving among other things a resolution to an old puzzle related to the beta function and a sufficient criteria for screening. We also extend our results to the  $SO(N_c)$  case where we present a candidate for the Wilson loop in the spinorial representation. Various aspects of this line of research are critically analyzed.

---

<sup>1</sup>c.h.badajoz@swansea.ac.uk

<sup>2</sup>c.nunez@swansea.ac.uk

<sup>3</sup>i.papadimitriou@swansea.ac.uk

# Contents

<b>1</b>	<b>Introduction</b>	<b>2</b>
<b>2</b>	<b>Comments on the Field Theory and String dual</b>	<b>3</b>
2.1	Field Theory aspects . . . . .	3
2.2	The String dual . . . . .	4
<b>3</b>	<b>Unified view of the type A and type N backgrounds</b>	<b>7</b>
<b>4</b>	<b>New (and old) solutions</b>	<b>11</b>
4.1	Exact solutions that were already known . . . . .	12
4.2	A new exact solution . . . . .	13
4.3	Classification of asymptotic solutions . . . . .	13
4.3.1	UV asymptotics . . . . .	13
4.3.2	IR asymptotics . . . . .	20
4.4	Solutions of the BPS equations as RG flows . . . . .	27
<b>5</b>	<b>Physics of the new solutions</b>	<b>31</b>
5.1	General Comments on the Field Theory . . . . .	31
5.2	Physics in the Infrared . . . . .	33
5.2.1	Enhancement of the flavor group . . . . .	33
5.2.2	String-like objects . . . . .	34
5.2.3	Domain-wall tensions . . . . .	37
5.2.4	Quartic coupling . . . . .	38
5.2.5	Screening as string breaking . . . . .	38
5.3	Physics in the Ultraviolet . . . . .	41
5.3.1	Beta function . . . . .	41
<b>6</b>	<b>The case of <math>SO(N_c)</math> gauge group</b>	<b>44</b>
6.1	Spinor Wilson loop . . . . .	47
<b>7</b>	<b>General comments, criticism and conclusions</b>	<b>48</b>
7.1	Conclusions . . . . .	52
<b>8</b>	<b>Acknowledgments</b>	<b>52</b>
<b>A</b>	<b>Appendix: Some aspects of the QFT</b>	<b>52</b>

<b>B Appendix: Expansions in integration constants</b>	<b>55</b>
B.1 Large $c_+$ expansion . . . . .	55
B.2 Small $c_+$ expansion . . . . .	56

# 1 Introduction

In the last decade, the AdS/CFT conjecture originally proposed by Maldacena [1] and refined in [2, 3] have shown to be one of the most powerful analytic tools to study strong coupling effects in gauge theories. The project of extending the original duality to theories with a renormalization-group flow was initiated in [4] and many different lines of research were proposed to compute non-perturbative effects in quantum field theories with small amount of Supersymmetry (SUSY).

In this paper we will focus on the type of set-ups called “wrapped branes models”. The first example of these models was presented by Witten in [5] and consists of a set of  $N_c$  D4 branes that wrap a circle with SUSY breaking periodicity conditions, giving at low energies an effective theory with a massless vector field. This String background may be thought as capturing strongly coupled aspects of a version of Yang-Mills theory completed at slightly higher energies by a set of extra states. This type of ideas have been applied to a variety of models, preserving different amounts of SUSY. For example in [6] a String background dual to an  $N = 2$  Super Yang-Mills theory was given. In the paper [7], the dual to a version of  $N = 1$  Super Yang-Mills was presented <sup>1</sup>.

The models described above involve a (large) number  $N_c$  of “color branes” usually wrapping calibrated cycles inside CY folds. In this paper we will focus on duals to field theories encoding the dynamics of adjoint fields in interaction with fundamental matter. Fundamental fields (quarks) are added, following the results of the paper [10], with “flavor branes”. These are branes sharing the Minkowski directions with the “color branes” and extended over non-compact calibrated manifolds inside the CY fold. For the case of wrapped brane set-ups dual to a version of  $N = 1$  SYM theory [7], the addition of flavors was first considered in the limit  $\frac{N_f}{N_c} \rightarrow 0$  (similar to the quenched approximation in the Lattice) in [11]. In the papers [9] and [12] the full dynamics of fundamentals was taken into account, by working in the so called Veneziano scaling, that is considering  $x = \frac{N_f}{N_c}$  fixed, in the large  $N_c$  limit. The field theory dual to the backgrounds in [9] and [12] is a version of  $N = 1$  SQCD on which we will elaborate below.

The rest of this paper is organized as follows: In Section 2, we will review the field theory and the String dual(s) presented in [9] and [12]. Then, in Section 3 we will present a unified treatment of the different string duals. This will allow us to systematically classify and find

---

<sup>1</sup>The solution in [7], is the one found in a 4-d gauged Supergravity context by the authors of [8]. There are actually a whole family of solutions dual to  $N=1$  SYM with an UV completion. See Section 8 of [9].

new solutions (done in Section 4). We will then dedicate Section 5 to the study of field theory aspects that can be read from the string dual; most notably we will provide a sufficient condition for screening of the Wilson and other loops, solve an old puzzle related to the beta function computation and study domain walls, Wilson, 't Hooft and dyon loops. In Section 6, we will present a dual version to the field theory described in Section 2 for the case of orthogonal gauge group  $SO(N_c)$  and present an object that can be associated with the Wilson loop in the spinor representation. Finally, in Section 7 we close with general criticism to this line of research and some conclusions. Various Appendixes with technical details complement our presentation, trying to make the main part of the paper more readable.

## 2 Comments on the Field Theory and String dual

### 2.1 Field Theory aspects

Let us start by commenting on the field theory at weak coupling. Before the addition of fundamental fields, this is a four dimensional theory preserving  $N = 1$  SUSY obtained via a twisted compactification of six dimensional Super Yang-Mills with sixteen supercharges. It is precisely the twisting in the compactification that preserves only four supercharges. The weakly coupled massless spectrum and multiplicities were studied in detail in [13]. The theory contains a massless vector multiplet plus a tower of massive chiral and massive vector multiplets, usually called “Kaluza-Klein (KK) modes”. The massive chiral superfields, denoted below by  $\Phi_k$  have masses (in units of the size of the inverse  $S^2$  radius) given by  $M_{\Phi_k}^2 = k(k+1)$  and degeneracy  $g = (2k+1)$ . The massive vector superfields, denoted below by  $V_k$ , have masses  $M_{V_k}^2 = k^2$  and degeneracy  $g = 4k$ .

Generically, the Lagrangian describing the weakly coupled theory (see [13] for the quadratic part of the Lagrangian), consists of a massless vector multiplet plus an infinite set of KK multiplets. Denoting the massless vector multiplet and its curvature by  $(V, \mathcal{W}_\alpha)$ , the massive vector multiplets by  $V_k$  (its curvature by  $W_k$ ) and massive chiral multiplets as  $\Phi_k$ , the action is

$$S = \int d^4\theta \sum_k \Phi_k^\dagger e^V \Phi_k + \mu_k |V_k|^2 + \int d^2\theta \left[ \mathcal{W}_\alpha \mathcal{W}^\alpha + \sum_k W_k W_k + \mu_k |\Phi_k|^2 + \mathcal{W}(\Phi_k, V_k) \right]. \quad (2.1)$$

On the basis of renormalizability, we will propose that the superpotential is at most cubic in the chirals, and we also expect that all the KK modes interact among themselves and that chirals interact with massive vectors via the kinetic term

$$\mathcal{W} = \sum_{ijk} z_{ijk} \Phi_i \Phi_j \Phi_k + \sum_k \hat{f}(\Phi_k) W_{k,\alpha} W_k^\alpha. \quad (2.2)$$

Now, suppose we want to couple this field theory to fundamental matter.

Addition of Flavors: We now introduce flavors as a pair of chiral multiplets  $Q, \bar{Q}$  transforming in the fundamental of  $SU(N_c)$ . The action will have the usual kinetic terms plus interactions between quarks and KK modes of the form

$$S_{Q,\bar{Q}} = \int d^4\theta (\bar{Q}^\dagger e^V \bar{Q} + Q^\dagger e^{-V} Q) + \int d^2\theta \sum_{p,i,j,a,b} \kappa_p^{ij} \bar{Q}^{a,i} \Phi_p^{ab} Q^{b,j}. \quad (2.3)$$

Here,  $a, b = 1, \dots, N_c$  are indexes in the fundamental and anti-fundamental of  $SU(N_c)$ , while  $i, j = 1, \dots, N_f$  belong to the fundamental and anti-fundamental of  $SU(N_f)$ . Notice that the interaction between the quark superfields and the KK fields is such that the  $SU(N_f)_V$  global symmetry is broken to  $U(1)^{N_f}$ . This reflects in the String solution via a smearing procedure that will be applied. With these interaction terms, it is possible to integrate out most of the KK modes at any given range of energy, although some may be light and should be kept in the action.

We can look for a configuration along the lines of what was proposed in the paper [12], that is, to search for a solution to the F term equations that is such that the cubic potential has no contribution. After integrating out the massive KK fields we find a low energy effective action of the form

$$S = S_{N=1 \text{ SQCD}} + W_{eff}, \quad (2.4)$$

with a superpotential given by (the color indexes of the quarks are contracted and suppressed)

$$W_{eff} \sim - \sum_{p,i,j} \frac{\kappa_{(p),ij}^2}{2\mu_p^2} (\bar{Q}_i Q_j)^2 \sim \frac{\kappa^2}{2\mu} M^2. \quad (2.5)$$

Let us now explain how this particular field theory, realized in a particular vacuum, is encoded in a String background.

## 2.2 The String dual

As it should be clear to the reader, the addition of flavors to a QFT using a dual String background is achieved by supplementing the putative closed string background with open string degrees of freedom [10].

One may decide to neglect the effects of pair creation of fundamental fields. This is analogous to neglect the deformation that the flavor branes should produce on the closed string background mentioned above-for more discussion see Section 7.

On the contrary, if the full quantum dynamics of fundamentals is to be considered, it should be encoded in a String background whose equations of motion are derived from the action

$$S = S_{type \text{ IIB/A}} + S_{branes}. \quad (2.6)$$

In this paper, we will follow the lead of [9] where the closed string background was taken to be the one of [7] supplemented by the dynamics of fundamentals. In the case at hand, the fundamentals are represented by a set of  $N_f$  D5 branes extended on a non-compact two-cycle of a CY3-fold. As mentioned above and explained in detail in [9] a smearing procedure is applied so that the eqs. of motion derived from (2.6) are ordinary differential eqs. From this perspective, this smearing is just a matter of technical convenience, for more discussion about the effects of the smearing, see Section 7.

To be concrete, we will choose coordinates  $x^M = [t, \vec{x}_3, \rho, \theta, \varphi, \tilde{\theta}, \tilde{\varphi}, \psi]$ . In the present situation, our eqs. of motion will describe two sets of branes: the  $N_c$  D5 color branes that wrap a compact calibrated two cycle inside a CY3-fold and  $N_f$  D5 flavor branes that extend along the same ‘Minkowski’ directions as the color branes wrapping a non-compact two manifold inside the CY3-fold. Besides, these flavor branes are smeared along the transversal four directions (hence avoiding dependencies on those four coordinates denoted by  $\theta, \tilde{\theta}, \varphi, \tilde{\varphi}$ ). The action from which the dynamics follows is (see [9] for a derivation)

$$S = \frac{1}{2\kappa_{(10)}^2} \int d^{10}x \sqrt{-g} \left[ R - \frac{1}{2} \partial_\mu \phi \partial^\mu \phi - \frac{1}{12} e^\phi F_{(3)}^2 \right] + \frac{T_5 N_f}{4\pi^2} \left( - \int_{\mathcal{M}_6} d^{10}x \sin \theta \sin \tilde{\theta} e^{\frac{\phi}{2}} \sqrt{-\hat{g}_{(6)}} + \int \text{Vol}(\mathcal{Y}_4) \wedge C_6 \right). \quad (2.7)$$

The authors of [9] proposed a configuration where the ‘topology’ of the metric was basically  $\mathbb{R}_{x^\mu}^{1,3} \times S_{\theta, \varphi}^2 \times \mathbb{R}_\rho \times S_{\tilde{\theta}, \tilde{\varphi}, \psi}^3$  (there is a fibration between  $S^2$  and  $S^3$  preserving four supercharges), a dilaton field  $\phi(\rho)$  depending on the radial coordinate and a RR three form that indicates the presence of sources via the Bianchi identity.

Two types of backgrounds have been proposed as possible solutions. In the paper [12], these solutions were referred to as type **A** and type **N** backgrounds and we will adopt that notation here. We will set the constants  $\alpha' = g_s = 1$  in the following. Then,  $(2\pi)^5 T_5 = 1$  and  $2\kappa_{10}^2 = (2\pi)^7$ .

The type **A** backgrounds: are believed to faithfully reproduce non-perturbative physics if the relation  $N_c \leq N_f$  is satisfied. The metric in the Einstein frame, the RR three form and the dilaton read

$$\begin{aligned} ds^2 &= e^{\frac{\phi(\rho)}{2}} \left[ dx_{1,3}^2 + 4Y(\rho) d\rho^2 + H(\rho) (d\theta^2 + \sin^2 \theta d\varphi^2) + G(\rho) (d\tilde{\theta}^2 + \sin^2 \tilde{\theta} d\tilde{\varphi}^2) \right. \\ &\quad \left. + Y(\rho) (d\psi + \cos \tilde{\theta} d\tilde{\varphi} + \cos \theta d\varphi)^2 \right], \\ F_{(3)} &= - \left[ \frac{N_c}{4} \sin \tilde{\theta} d\tilde{\theta} \wedge d\tilde{\varphi} + \frac{N_f - N_c}{4} \sin \theta d\theta \wedge d\varphi \right] \wedge (d\psi + \cos \tilde{\theta} d\tilde{\varphi} + \cos \theta d\varphi), \\ \phi &= \phi(\rho), \end{aligned} \quad (2.8)$$

The functions  $H(\rho)$ ,  $G(\rho)$ ,  $Y(\rho)$ ,  $\phi(\rho)$  satisfy a set of BPS equations that as usual solve all

the Euler-Lagrange equations derived from (2.7). These first order non-linear equations read,

$$H' = \frac{1}{2}(N_c - N_f) + 2Y, \quad (2.9)$$

$$G' = -\frac{N_c}{2} + 2Y, \quad (2.10)$$

$$Y' = -\frac{1}{2}(N_f - N_c)\frac{Y}{H} - \frac{N_c Y}{2G} - 2Y^2 \left( \frac{1}{H} + \frac{1}{G} \right) + 4Y, \quad (2.11)$$

$$\phi' = -\frac{(N_c - N_f)}{4H} + \frac{N_c}{4G}. \quad (2.12)$$

In the paper [12] these equations were solved (partly analytically and partly numerically). The interesting finding was the existence of three types of solutions having very different IR behavior<sup>2</sup>. They were called respectively type I, type II, and type III. The three types of IR solutions connect smoothly with solutions at very large values of  $\rho$  (the UV of the dual QFT). The difference between the three types of UVs is in very suppressed terms  $\mathcal{O}(e^{-4\rho})$ , that can be interpreted as different VEVs for operators of dimension six in the dual QFT. See [12] for a clear explanation of this issue. For completeness, we will re-analyze these solutions in Section 4, also finding new behaviors.

The type N backgrounds: we know much less about these backgrounds. The only available solution was originally found (in part analytically and in part numerically) in [9]. The configuration of the dilaton, metric (in Einstein frame) and RR three form are given by

$$\begin{aligned} ds^2 &= e^{\frac{\phi(\rho)}{2}} \left[ dx_{1,3}^2 + e^{2k(\rho)} d\rho^2 + e^{2h(\rho)} (d\theta^2 + \sin^2 \theta d\varphi^2) + \right. \\ &\quad \left. + \frac{e^{2g(\rho)}}{4} ((\tilde{\omega}_1 + a(\rho)d\theta)^2 + (\tilde{\omega}_2 - a(\rho)\sin\theta d\varphi)^2) + \frac{e^{2k(\rho)}}{4} (\tilde{\omega}_3 + \cos\theta d\varphi)^2 \right], \\ F_{(3)} &= \frac{N_c}{4} \left[ -(\tilde{\omega}_1 + b(\rho)d\theta) \wedge (\tilde{\omega}_2 - b(\rho)\sin\theta d\varphi) \wedge (\tilde{\omega}_3 + \cos\theta d\varphi) + \right. \\ &\quad \left. b'd\rho \wedge (-d\theta \wedge \tilde{\omega}_1 + \sin\theta d\varphi \wedge \tilde{\omega}_2) + (1 - b(\rho)^2) \sin\theta d\theta \wedge d\varphi \wedge \tilde{\omega}_3 \right] \\ &\quad - \frac{N_f}{4} \sin\theta d\theta \wedge d\varphi \wedge (d\psi + \cos\tilde{\theta} d\tilde{\varphi}), \end{aligned} \quad (2.13)$$

where  $\tilde{\omega}_i$  are the left-invariant forms of  $SU(2)$

$$\begin{aligned} \tilde{\omega}_1 &= \cos\psi d\tilde{\theta} + \sin\psi \sin\tilde{\theta} d\tilde{\varphi}, \\ \tilde{\omega}_2 &= -\sin\psi d\tilde{\theta} + \cos\psi \sin\tilde{\theta} d\tilde{\varphi}, \\ \tilde{\omega}_3 &= d\psi + \cos\tilde{\theta} d\tilde{\varphi}, \end{aligned} \quad (2.14)$$

---

<sup>2</sup>The word IR (infrared) actually refers to the IR of the dual QFT. On the string side we mean a solution for small values of the radial coordinate  $\rho$ . Here and in the rest of the paper we will adopt this terminology.

and the BPS equations defining the functions  $h, g, k, a, b, \phi$  as presented in Appendix B of [9] and appear quite involved; we will not quote them here because in the following sections we will present them in a unified way with the type **A** BPS equations written above eqs.(2.9)-(2.12).

It was argued in [9] and [12] that the type **N** backgrounds faithfully capture non-perturbative physics for any value of the number of flavors  $N_f > 0$ . Also, it was proposed that backgrounds of type **N** describe vacua of the theory where the gaugino condensate is non-zero, while type **A** backgrounds describe vacua with  $\langle \lambda\lambda \rangle = 0$ . By the Konishi anomaly relation, this seems to indicate that the meson superfield should vanish in type **A** backgrounds whilst being non-zero in type **N** solutions.

Many checks or predictions about aspects of the QFT in eq. (2.4) using the type **A** and type **N** solutions have been presented in [9, 12, 14].

Let us then start with the main part of this paper, describing in a unified fashion the type **A**, **N** backgrounds mentioned above.

### 3 Unified view of the type **A** and type **N** backgrounds

It is clear that the type **A** configurations are a special case of the type **N** ones. Indeed, if the functions  $a(\rho) = b(\rho) = 0$ , then there is no difference between the type **A** and type **N** backgrounds. Hence, the fact that the BPS equations for type **A** solutions read like in eqs. (2.9)-(2.12) suggests that there should be some variables where the type **N** first order eqs. will look similarly nice.

So, we start by rewriting the type **N** ansatz (2.13) using a more symmetric parametrization as

$$ds^2 = e^{\phi(\rho)/2} \left\{ dx_{1,3}^2 + Y(\rho) (4d\rho^2 + (\omega_3 + \tilde{\omega}_3)^2) + \frac{1}{2}P(\rho) \sinh \tau(\rho) (\omega_1 \tilde{\omega}_1 - \omega_2 \tilde{\omega}_2) \right. \\ \left. + \frac{1}{4} (P(\rho) \cosh \tau(\rho) + Q(\rho)) (\omega_1^2 + \omega_2^2) + \frac{1}{4} (P(\rho) \cosh \tau(\rho) - Q(\rho)) (\tilde{\omega}_1^2 + \tilde{\omega}_2^2) \right\}, \quad (3.1)$$

$$F_{(3)} = -d \{ \sigma(\rho) (\omega_1 \wedge \tilde{\omega}_1 - \omega_2 \wedge \tilde{\omega}_2) \} - \left( \frac{N_f - N_c}{4} \omega_1 \wedge \omega_2 + \frac{N_c}{4} \tilde{\omega}_1 \wedge \tilde{\omega}_2 \right) \wedge (\omega_3 + \tilde{\omega}_3), \quad (3.2)$$

where we have defined, in addition to the left invariant forms (2.14),  $\omega_1 = d\theta$ ,  $\omega_2 = \sin \theta d\varphi$ ,  $\omega_3 = \cos \theta d\varphi$ . The relation between these new variables and the functions originally parametrizing the type **N** backgrounds in eq. (2.13) is

$$e^{2h} = \frac{1}{4} \left( \frac{P^2 - Q^2}{P \cosh \tau - Q} \right), \quad e^{2g} = P \cosh \tau - Q, \quad e^{2k} = 4Y, \quad a = \frac{P \sinh \tau}{P \cosh \tau - Q}, \quad b = \frac{\sigma}{N_c}. \quad (3.3)$$



Evaluating the action (2.7) on the ansatz (3.1)-(3.2) we obtain the one-dimensional effective action

$$(2\pi)^4 \text{Vol}(\mathbb{R}^{1,3})^{-1} S \equiv S_{1d} = \int d\rho \mathcal{L}, \quad (3.4)$$

where the 1-d effective Lagrangian is,

$$\mathcal{L} = \frac{\Phi}{4\sqrt{Y}} \left( \left( \frac{\Phi'}{\Phi} \right)^2 - \frac{1}{4} \left( \frac{Y'}{Y} \right)^2 - \frac{1}{2} \left( \frac{P' + Q'}{P + Q} \right)^2 - \frac{1}{2} \left( \frac{P' - Q'}{P - Q} \right)^2 - \left( \frac{P^2 \tau'^2 + \sigma'^2}{P^2 - Q^2} \right) \right) - V, \quad (3.5)$$

we have defined  $\Phi \equiv (P^2 - Q^2)Y^{1/2}e^{2\phi}$ . The potential is given by

$$V = \frac{\Phi}{(P^2 - Q^2)\sqrt{Y}} \left\{ \frac{(8YP)^2 + ((2N_c - N_f)P \cosh \tau + N_f Q - 2\sigma P \sinh \tau)^2}{2(P^2 - Q^2)} - 4Y(4P \cosh \tau - N_f + 4Y) + P^2 \sinh^2 \tau + N_c(N_f - N_c) + \sigma^2 \right\}, \quad (3.6)$$

At this point a word of caution is in order. Namely, the solutions of the Euler-Lagrange equations of the one dimensional effective action (3.5) are *not* necessarily solutions of the Type IIB plus branes equations of motion. This is because we have obtained (3.5) by inserting the ansatz in the supergravity action and not in the supergravity equations of motion. It turns out, however, that there is a special class of solutions of the Euler-Lagrange equations of (3.5) that satisfy precisely the BPS equations of the supersymmetric type **A** and type **N** backgrounds. Since it is known that these BPS equations imply the second order supergravity plus branes equations [9] (for a general proof see [15]), it follows that this particular class of solutions of (3.5) are actually solutions of the second order supergravity plus branes equations. The reason why we consider the one dimensional effective action (3.5) at all is that it naturally leads to a ‘superpotential’ for the BPS equations of both type **A** and type **N** backgrounds.

It will often be convenient to write  $P, Q$ , in terms of further new variables  $H, G$  as  $P = 2(H + G)$ ,  $Q = 2(H - G)$ . Note that for the type **A** backgrounds, obtained by setting  $\tau = \sigma = 0$ , the variables  $H, G$ , defined as above coincide with those in (2.8)-(2.12).

From the effective action (3.5) then we compute the canonical momenta

$$\begin{aligned} \pi_H &= \frac{\partial \mathcal{L}}{\partial H'} = -\frac{1}{4} \Phi Y^{-1/2} \frac{H'}{H^2}, & \pi_G &= \frac{\partial \mathcal{L}}{\partial G'} = -\frac{1}{4} \Phi Y^{-1/2} \frac{G'}{G^2}, \\ \pi_Y &= \frac{\partial \mathcal{L}}{\partial Y'} = -\frac{1}{8} \Phi Y^{-5/2} Y', & \pi_\Phi &= \frac{\partial \mathcal{L}}{\partial \Phi'} = \frac{1}{2} Y^{-1/2} \frac{\Phi'}{\Phi}, \\ \pi_\tau &= \frac{\partial \mathcal{L}}{\partial \tau'} = -\frac{1}{2} \Phi Y^{-1/2} \frac{P^2 \tau'}{P^2 - Q^2}, & \pi_\sigma &= \frac{\partial \mathcal{L}}{\partial \sigma'} = -\frac{1}{2} \Phi Y^{-1/2} \frac{\sigma'}{P^2 - Q^2}, \end{aligned} \quad (3.7)$$

as well as the Hamiltonian

$$\begin{aligned} \mathcal{H} &= H' \pi_H + G' \pi_G + Y' \pi_Y + \Phi' \pi_\Phi + \tau' \pi_\tau + \sigma' \pi_\sigma - \mathcal{L} \\ &= Y^{1/2} \Phi^{-1} \left( \Phi^2 \pi_\phi^2 - 2H^2 \pi_H^2 - 2G^2 \pi_G^2 - 4Y^2 \pi_Y^2 - (P^2 - Q^2)(P^{-2} \pi_\tau^2 + \pi_\sigma^2) \right) + V. \end{aligned} \quad (3.8)$$

Using Hamilton-Jacobi theory we can obtain the canonical momenta as derivatives of Hamilton's principal function,  $\mathcal{S}$ , as

$$\pi_H = \frac{\partial \mathcal{S}}{\partial H}, \quad \pi_G = \frac{\partial \mathcal{S}}{\partial G}, \quad \pi_Y = \frac{\partial \mathcal{S}}{\partial Y}, \quad \pi_\Phi = \frac{\partial \mathcal{S}}{\partial \Phi}, \quad \pi_\tau = \frac{\partial \mathcal{S}}{\partial \tau}, \quad \pi_\sigma = \frac{\partial \mathcal{S}}{\partial \sigma}, \quad (3.9)$$

where  $\mathcal{S}$  is a solution of the Hamilton-Jacobi equation

$$\mathcal{H}\left(H, G, Y, \Phi, \tau, \sigma; \frac{\partial \mathcal{S}}{\partial H}, \frac{\partial \mathcal{S}}{\partial G}, \frac{\partial \mathcal{S}}{\partial Y}, \frac{\partial \mathcal{S}}{\partial \Phi}, \frac{\partial \mathcal{S}}{\partial \tau}, \frac{\partial \mathcal{S}}{\partial \sigma}\right) = 0. \quad (3.10)$$

A particular solution to this equation is,

$$\begin{aligned} \mathcal{S} = \frac{\Phi}{16\sqrt{Y}} & \left\{ (8Y - N_f) \left( \frac{1}{H} + \frac{1}{G} \right) + ((2N_c - N_f) \cosh \tau - 2\sigma \sinh \tau) \left( \frac{1}{H} - \frac{1}{G} \right) \right. \\ & \left. + 16 \cosh \tau \right\}. \end{aligned} \quad (3.11)$$

By equating the canonical momenta (3.7) to the corresponding ones obtained from (3.9), this incomplete integral (cf. superpotential) leads to a system of first order equations, whose solutions *automatically* satisfy the second order equations following from the Lagrangian (3.5) – but, we stress again, *not* necessarily the Type IIB plus branes equations –

$$\begin{aligned} H' &= \frac{1}{4} \{ 8Y - N_f + ((2N_c - N_f) \cosh \tau - 2\sigma \sinh \tau) \}, \\ G' &= \frac{1}{4} \{ 8Y - N_f - ((2N_c - N_f) \cosh \tau - 2\sigma \sinh \tau) \}, \\ Y' &= \frac{Y}{4} \left\{ - (8Y + N_f) \left( \frac{1}{H} + \frac{1}{G} \right) + ((2N_c - N_f) \cosh \tau - 2\sigma \sinh \tau) \left( \frac{1}{H} - \frac{1}{G} \right) \right. \\ & \quad \left. + 16 \cosh \tau \right\}, \\ \Phi' &= \frac{1}{8} \Phi \left\{ (8Y - N_f) \left( \frac{1}{H} + \frac{1}{G} \right) + ((2N_c - N_f) \cosh \tau - 2\sigma \sinh \tau) \left( \frac{1}{H} - \frac{1}{G} \right) \right. \\ & \quad \left. + 16 \cosh \tau \right\}, \\ \tau' &= \frac{1}{2(H+G)^2} \{ (H-G) ((2N_c - N_f) \sinh \tau - 2\sigma \cosh \tau) - 16HG \sinh \tau \}, \\ \sigma' &= -4(H-G) \sinh \tau. \end{aligned} \quad (3.12)$$

Notice that for  $\tau = \sigma = 0$ , this system of first order equations reduces to the BPS eqs. (2.9)-(2.12), as claimed above. However, as they stand, (3.12) are not quite the same as the type **N** BPS equations.

Let us next introduce the function

$$\omega \equiv \sigma - \tanh \tau \left( Q + \frac{2N_c - N_f}{2} \right). \quad (3.13)$$

The first order equations (3.12) can then be rearranged in the form

$$\begin{aligned} P' &= 8Y - N_f, \\ \partial_\rho \left( \frac{Q}{\cosh \tau} \right) &= \frac{(2N_c - N_f)}{\cosh^2 \tau} - \frac{2\omega}{P^2} (P^2 - Q^2) \tanh \tau, \\ \partial_\rho \log \left( \frac{\Phi}{\sqrt{P^2 - Q^2}} \right) &= 2 \cosh \tau, \\ \partial_\rho \log \left( \frac{\Phi}{\sqrt{Y}} \right) &= \frac{16YP}{P^2 - Q^2}, \\ \tau' + 2 \sinh \tau &= -\frac{2Q \cosh \tau}{P^2} \omega, \\ \omega' &= \frac{2\omega}{P^2 \cosh \tau} \left( P^2 \sinh^2 \tau + Q \left( Q + \frac{2N_c - N_f}{2} \right) \right). \end{aligned} \quad (3.14)$$

It is clear from this form of the first order equations that the algebraic constraint

$$\omega = 0 \Leftrightarrow \sigma = \tanh \tau \left( Q + \frac{2N_c - N_f}{2} \right), \quad (3.15)$$

is consistent with the equations and therefore defines a subclass of solutions, eliminating one integration constant. It can be shown that the first order equations (3.12), *together* with the algebraic constraint  $\omega = 0$ , are equivalent to the BPS equations for the type **N** backgrounds, derived in Appendix B of [9]. The subclass of solutions of eqs. (3.12) that satisfy the constraint (3.15) then correspond to supersymmetric solutions. As stressed above, in the case  $\sigma = \tau = 0$ , eqs. (3.12) reduce to the type **A** BPS eqs. (2.9)-(2.12), and so these solutions correspond to the supersymmetric type **A** solutions. The constraint in eq. (3.15) is automatically satisfied in this case.

The form of eqs. (3.14) makes it manifest that imposing the constraint in eq. (3.15) leads to a dramatic simplification. Firstly, the equation for  $\omega$  is trivially satisfied, while the equation for  $\tau$  decouples and gives

$$\sinh \tau = \frac{1}{\sinh(2(\rho - \rho_o))}. \quad (3.16)$$

Given this, one can then integrate the equations for  $Q$  and  $\Phi$  to obtain

$$Q = \left( Q_o + \frac{2N_c - N_f}{2} \right) \cosh \tau + \frac{2N_c - N_f}{2} (2\rho \cosh \tau - 1), \quad (3.17)$$

$$e^{4(\phi-\phi_o)} = \frac{\sinh^2(2\rho_o)}{(P^2 - Q^2)Y \sinh^2 \tau}. \quad (3.18)$$

Note that both  $Q$  and the dilaton are given *algebraically* in terms of the rest of the functions parametrizing the backgrounds. Here  $\rho_o$ ,  $Q_o$  and  $\phi_o$  are constants of integration and we have chosen the integration constant in (3.18) such that it admits a smooth limit as  $\rho_o \rightarrow -\infty$  (this limit gives  $\tau = \sigma = 0$  and so corresponds to the type **A** backgrounds). Finally,  $Y$  is determined in terms of  $P$  as

$$Y = \frac{1}{8}(P' + N_f), \quad (3.19)$$

while the only remaining unknown, the function  $P$ , then satisfies the decoupled second order equation

$$P'' + (P' + N_f) \left( \frac{P' + Q' + 2N_f}{P - Q} + \frac{P' - Q' + 2N_f}{P + Q} - 4 \cosh \tau \right) = 0. \quad (3.20)$$

We will refer to this equation as the ‘master’ equation, since once we have a solution of (3.20) all other functions are determined via (3.16)-(3.19).

Some comments are due here. First, note that the number of integration constants is indeed as expected. Namely, (3.12) are six first order equations for six variables and so we expect six integration constants. However, the algebraic constraint (3.15) eliminates one of the integration constants. The five integration constants are then  $\rho_o$ ,  $Q_o$ ,  $\phi_o$  plus the two integration constants coming from the second order equation (3.20). As we shall discuss in Section 5, the integration constant  $\rho_o$  is related to the gaugino condensate VEV. In particular, a finite value of  $\rho_o$  corresponds to the type **N** backgrounds, while the limit  $\rho_o \rightarrow -\infty$  gives the type **A** backgrounds. As we will state below, the type **N/A** backgrounds are dual to the field theory in vacua with/without gaugino bilinear VEV.

## 4 New (and old) solutions

In this section we will present solutions to the first order eqs. (3.12), which also satisfy the constraint (3.15). As we have just discussed, these correspond to the supersymmetric type **A** and type **N** solutions. We will write the solutions in terms of the new variables  $P, Q, Y, \tau, \sigma$  and  $e^{2\phi}$  as given in eqs. (3.3), with the values of  $Q, Y, \tau, \sigma$  and  $e^{2\phi}$  being read from eqs. (3.16)-(3.19) and obtained after solving for  $P$  in eq. (3.20). In order to facilitate easy comparison with the earlier literature, however, we will often write the solutions in the original variables too.

We will present first some solutions that were already known, just to convince the reader that our general treatment of Section 3 is correct (and convenient). Then we will present a new exact solution (that we have found for the particular case of  $N_f = 2N_c$ ). Finally, we will present more general solutions that we know only as series expansions for large and small

values of the radial coordinate, or perturbatively in certain integration constants. They will describe the UV and IR physics of the dual field theory.

To close the section, we will present a study of the first order eqs. (3.12) as a dynamical system. This will give further insight into the dual theory dynamics.

## 4.1 Exact solutions that were already known

Unflavored  $N_f = 0$  solutions:

For  $N_f = 0$  an exact solution of (3.20) is the well known unflavored solution [7, 8], which for  $\rho_o > -\infty$  (that is type **N** solution) takes the form

$$P = 2N_c(\rho - \rho_o), \quad Q_o = -N_c - 2N_c\rho_o, \quad \rho \geq \rho_o, \quad (4.1)$$

or equivalently

$$\begin{aligned} e^{2h} &= \frac{N_c}{8 \sinh^2(2(\rho - \rho_o))} \left[ 1 - 8(\rho - \rho_o)^2 - \cosh(4(\rho - \rho_o)) - 4(\rho - \rho_o) \sinh(4(\rho - \rho_o)) \right], \\ e^{2g} = e^{2k} &= N_c, \quad a(\rho) = b(\rho) = \frac{2(\rho - \rho_o)}{\sinh(2(\rho - \rho_o))}, \\ e^{-2\phi} &= \frac{2e^h}{\sinh(2(\rho - \rho_o))}. \end{aligned} \quad (4.2)$$

This is a smooth solution. On the other hand, for  $\rho_o \rightarrow -\infty$  (type **A** solution), we have instead

$$P = 2N_c(\rho - \rho_*), \quad \rho_* \equiv -\frac{1}{2N_c} \left( Q_o + \frac{N_c}{2} \right), \quad \rho \geq \rho_*, \quad (4.3)$$

that gives a background like the one in eq. (2.8) with,

$$H = N_c(\rho - \rho_*), \quad G = \frac{N_c}{4}, \quad Y = \frac{N_c}{4}, \quad e^{4(\phi - \phi_o)} = \frac{e^{4\rho}}{N_c^3(\rho - \rho_*)}. \quad (4.4)$$

This solution has a singularity at  $\rho = \rho_*$ .

The case  $N_f = 2N_c$ :

For  $N_f = 2N_c$  an exact type **A** ( $\rho_o \rightarrow -\infty$ ) solution of (3.20) is known [9, 12]<sup>3</sup>. It reads

$$P = N_c + \sqrt{N_c^2 + Q_o^2}, \quad Q = Q_o \equiv 4N_c \frac{(2 - \xi)}{\xi(4 - \xi)}, \quad 0 < \xi < 4, \quad (4.5)$$

or in terms of the background variables described in eq. (2.8)

$$H = \frac{N_c}{\xi}, \quad G = \frac{N_c}{4 - \xi}, \quad Y = \frac{N_c}{4}, \quad e^{4(\phi - \phi_o)} = \frac{\xi(4 - \xi)e^{4\rho}}{4N_c^3}. \quad (4.6)$$

---

<sup>3</sup>Another exact solution valid for any  $\rho_o > -\infty$  is  $P = -N_c \cosh \tau$ ,  $Q = \pm \sqrt{3}N_c \cosh \tau$ , but it is clearly singular. However, it might be useful in finding a related non-singular solution.

## 4.2 A new exact solution

Here we present a new one-parameter family of type **A** solutions for the case  $N_f = 2N_c$ . This solution is a one-parameter deformation of (4.5) for  $\xi = 4/3$  or  $\xi = 8/3$ , the two values of  $\xi$  being interchanged under Seiberg duality (Seiberg duality in these backgrounds corresponds to the transformation  $P \rightarrow P$ ,  $Q \rightarrow -Q$ ,  $\tau \rightarrow \tau$ ,  $Y \rightarrow Y$ ,  $N_c \rightarrow N_f - N_c$ . See (5.3) below).

The new solution of eq. (3.20), with  $\rho_o \rightarrow -\infty$  (type **A**), takes the form

$$P = \frac{9N_c}{4} + c_+ e^{4\rho/3}, \quad c_+ > 0, \quad Q = \pm \frac{3N_c}{4}, \quad (4.7)$$

or, in terms of the functions in the background described in eq. (2.8),

$$H = \frac{N_c}{16}(9 \pm 3) + \frac{c_+}{4}e^{4\rho/3}, \quad G = \frac{N_c}{16}(9 \mp 3) + \frac{c_+}{4}e^{4\rho/3}, \quad Y = \frac{N_c}{4} + \frac{c_+}{6}e^{4\rho/3},$$

$$e^{4(\phi-\phi_o)} = \frac{6e^{4\rho}}{(3N_c + c_+ e^{4\rho/3}) \left( \frac{3N_c}{2} + c_+ e^{4\rho/3} \right)^2}, \quad (4.8)$$

where  $c_+$  is an arbitrary positive constant. It can be seen that this solution approaches in the IR ( $\rho \rightarrow -\infty$ ) the solution in eq. (4.5) for the values  $(\xi = \frac{4}{3}, \frac{8}{3})$ , while it drastically differs from (4.5) in the UV ( $\rho \rightarrow \infty$ ). The new solution leads to an asymptotic UV geometry of the form  $\mathbb{R}^{1,3} \times M_6$ , where  $M_6$  is the conifold. Such UV asymptotics was anticipated in [16] and the solution (4.7) is the first exact example possessing this UV behavior. The physics of this and other solutions with similar UV behavior will be analyzed in Section 5. Finally, let us mention in passing that it may be possible to find solutions like the ones above where a black hole (non-extremal deformation) is introduced.

## 4.3 Classification of asymptotic solutions

Here we classify the asymptotic solutions of the ‘master’ equation (3.20), both in the UV ( $\rho \rightarrow \infty$ ) and in the IR ( $\rho \rightarrow \rho_{IR}$ ), where  $\rho_{IR} \geq \rho_o$  is the minimum value of the radial coordinate in the background geometry <sup>4</sup>. Note that for the type **A** backgrounds  $\rho_o \rightarrow -\infty$  and so it is possible that  $\rho_{IR} \rightarrow -\infty$ . For the type **N** backgrounds, however, we have instead  $\rho_{IR} \geq \rho_o > -\infty$ .

### 4.3.1 UV asymptotics

Let us start by discussing first the UV behavior of solutions to eq. (3.20). As we have already mentioned, by UV we mean the asymptotic behavior of the solutions as  $\rho \rightarrow \infty$  <sup>5</sup>. The first

---

<sup>4</sup>We stress again that the words UV-ultraviolet- and IR-infrared- make reference to the high and low energies (compared to  $\Lambda_{QCD}$ ) in the dual field theory. Here we will use these terms indistinctly with large and small  $\rho$  asymptotics.

<sup>5</sup>Note that this excludes solutions that describe a Landau pole in the dual theory, where the gauge coupling diverges at a finite value of the radial coordinate,  $\rho$ .

observation one can now make is that as  $\rho \rightarrow \infty$ ,  $\tau \rightarrow 0$  (see (3.16)) and so the leading asymptotic behavior of the type **A** and type **N** solutions of (3.20) is the same. This is indeed expected from the holographic interpretation of the type **N** backgrounds, which are believed to describe the same theory as the one described by the type **A** backgrounds, but with a non-zero VEV for the gaugino bilinear,  $\langle \lambda\lambda \rangle \neq 0$ . At energies much above the energy scale set by the gaugino condensate the two types of solutions must be identical.

Secondly, from (3.17) we see that  $Q \rightarrow \pm\infty$  as  $\rho \rightarrow \infty$  respectively for  $N_f < 2N_c$  or  $N_f > 2N_c$ , while it goes to a constant for  $N_f = 2N_c$ . Moreover, from (3.18) follows that both  $(P + Q)$  and  $(P - Q)$  should remain positive as  $\rho \rightarrow \infty$ , which requires that  $P \rightarrow +\infty$  as  $\rho \rightarrow \infty$  for  $N_f \neq 2N_c$ , while it can either asymptote to a finite constant or to infinity for  $N_f = 2N_c$ .

To summarize then, for  $N_f \neq 2N_c$  one needs to look for asymptotic solutions of (3.20) with  $P \rightarrow +\infty$  as  $\rho \rightarrow +\infty$ , while for  $N_f = 2N_c$  one should look for solutions such that  $P \rightarrow +\infty$  or  $P \rightarrow \text{constant}$  as  $\rho \rightarrow +\infty$ . One then finds that there are two classes of qualitatively different asymptotic solutions. The first class consists of solutions where  $P$  behaves linearly with  $\rho$  as  $\rho \rightarrow +\infty$ , with the coefficient dependent on the relative values of  $N_f$  and  $N_c$ . These asymptotic solutions have been studied in [9, 12]. The second class consists of solutions for which  $P \sim c_+ e^{4\rho/3}$  as  $\rho \rightarrow +\infty$ , where  $c_+$  is an arbitrary constant. This type of UV asymptotics was anticipated in the *flavorless* case in the paper [9], where it was pointed out (see Appendix B of [9]) that the geometry in this case asymptotes to four dimensional Minkowski spacetime times the conifold for the type **A** backgrounds ( $\rho_o \rightarrow -\infty$ ), or times the deformed conifold for the type **N** backgrounds ( $\rho_o > -\infty$ ).

Given this general classification of the UV asymptotics, which we have summarized in Table 1, we are now ready to construct explicitly the corresponding asymptotic solutions.

### **Class I:**

As we mentioned above, this class consists of solutions where  $P$  grows at most linearly with  $\rho$  as  $\rho \rightarrow +\infty$ . Since  $\cosh \tau = (1 + e^{-4(\rho - \rho_o)}) / (1 - e^{-4(\rho - \rho_o)}) = 1 + \mathcal{O}(e^{-4\rho})$ , it follows that unless one includes exponentially suppressed terms, the class I UV expansions of the type **A** and type **N** backgrounds are identical. Namely,

(i)  $N_f < 2N_c$ :

$$P = Q + N_c \left( 1 + \frac{N_f}{4Q} + \frac{N_f(N_f - 2N_c)}{8Q^2} + \frac{N_f(16N_c^2 - 19N_cN_f + 5N_f^2)}{32Q^3} + \mathcal{O}(Q^{-4}) \right). \quad (4.9)$$

$N_f$	I	II
$< 2N_c$	$P \sim Q \sim  2N_c - N_f  \rho$ $e^{2h} \sim \frac{1}{2} (2N_c - N_f) \rho$ $e^{2g} \sim N_c$ $Y \sim \frac{N_c}{4}$ $e^{4(\phi-\phi_o)} \sim \frac{e^{4(\rho-\rho_o)} \sinh^2(2\rho_o)}{2N_c^2 (2N_c - N_f) \rho}$ $a \sim \frac{2}{N_c} (2N_c - N_f) e^{-2(\rho-\rho_o)} \rho$	
$> 2N_c$	$P \sim -Q \sim  2N_c - N_f  \rho$ $e^{2h} \sim \frac{1}{4} (N_f - N_c)$ $e^{2g} \sim \frac{1}{2} (N_f - 2N_c) \rho$ $Y \sim \frac{1}{4} (N_f - N_c)$ $e^{4(\phi-\phi_o)} \sim \frac{e^{4(\rho-\rho_o)} \sinh^2(2\rho_o)}{2(N_c - N_f)^2 (N_f - 2N_c) \rho}$ $a \sim e^{-2(\rho-\rho_o)} \rho$	$P \sim c_+ e^{4\rho/3}$ $e^{2h} \sim \frac{1}{4} c_+ e^{4\rho/3}$ $e^{2g} \sim c_+ e^{4\rho/3}$ $Y \sim \frac{1}{6} c_+ e^{4\rho/3}$ $e^{4(\phi-\phi_o)} \sim 1$ $a \sim 2e^{-2(\rho-\rho_o)}$
$= 2N_c$	$P \sim N_c + \sqrt{N_c^2 + Q_o^2} \sim \frac{8N_c}{(4-\xi)\xi}$ $e^{2h} \sim \frac{N_c}{\xi}$ $e^{2g} \sim \frac{4N_c}{4-\xi}$ $Y \sim \frac{N_c}{4}$ $e^{4(\phi-\phi_o)} \sim e^{4(\rho-\rho_o)} \sinh^2(2\rho_o) \frac{(4-\xi)\xi}{16N_c^3}$ $a \sim \frac{4}{\xi} e^{-2(\rho-\rho_o)}$	

Table 1: The two classes of leading UV behaviors.



Using (3.16)-(3.19) this leads to

$$\begin{aligned}
e^{2h} &= \left(N_c - \frac{N_f}{2}\right) \rho + \frac{1}{4} (N_c + 2Q_o) + \frac{N_c N_f}{(32N_c - 16N_f) \rho} \\
&\quad + \frac{N_c N_f (-2N_c + N_f - 2Q_o)}{32 (-2N_c + N_f)^2 \rho^2} + \mathcal{O}(\rho^{-3}), \\
e^{2g} &= N_c + \frac{N_c N_f}{(8N_c - 4N_f) \rho} + \frac{N_c N_f (-2N_c + N_f - 2Q_o)}{8 (-2N_c + N_f)^2 \rho^2} + \mathcal{O}(\rho^{-3}), \\
Y &= \frac{N_c}{4} - \frac{N_c N_f}{(64N_c - 32N_f) \rho^2} + \frac{N_c N_f (2N_c - N_f + 2Q_o)}{32 (-2N_c + N_f)^2 \rho^3} + \mathcal{O}(\rho^{-4}), \\
e^{4(\phi - \phi_o)} &= e^{4(\rho - \rho_o)} \left( \frac{1}{(4N_c^3 - 2N_c^2 N_f) \rho} - \frac{2N_c + N_f + 4Q_o}{8 (N_c^2 (-2N_c + N_f)^2) \rho^2} + \mathcal{O}(\rho^{-3}) \right) \\
&\quad \times \sinh^2(2\rho_o), \\
a &= e^{-2(\rho - \rho_o)} \left( \left(4 - \frac{2N_f}{N_c}\right) \rho + \frac{4N_c - N_f + 4Q_o}{2N_c} + \frac{4N_c N_f - N_f^2}{(16N_c^2 - 8N_c N_f) \rho} \right. \\
&\quad \left. + \mathcal{O}(\rho^{-2}) \right). \quad (4.10)
\end{aligned}$$

It is now obvious that these expansions describe in a unified way both the type **A** UV expansions in eq. (3.5) in [12], as well as the type **N** UV expansions in eqs. (4.18)-(4.19) in [9].

(ii)  $N_f > 2N_c$ :

$$\begin{aligned}
P = -Q + (N_f - N_c) &\left( 1 - \frac{N_f}{4Q} - \frac{N_f(N_f - 2N_c)}{8Q^2} - \frac{N_f(16N_c^2 - 13N_c N_f + 2N_f^2)}{32Q^3} \right. \\
&\quad \left. + \mathcal{O}(Q^{-4}) \right). \quad (4.11)
\end{aligned}$$

This expansion is related to the expansion (4.9) by the Seiberg duality transformation

$Q \rightarrow -Q$ ,  $N_c \rightarrow N_f - N_c$ . Using again (3.16)-(3.19) this leads to the expansions

$$\begin{aligned}
e^{2h} &= \frac{1}{4}(-N_c + N_f) + \frac{(N_c - N_f)N_f}{(32N_c - 16N_f)\rho} - \frac{(N_c - N_f)N_f(2N_c - N_f + 2Q_o)}{32(-2N_c + N_f)^2\rho^2} \\
&\quad + \mathcal{O}(\rho^{-3}), \\
e^{2g} &= \frac{1}{2}(-2N_c + N_f)\rho + \frac{1}{4}(-N_c + N_f - 2Q_o) + \frac{(N_c - N_f)N_f}{(32N_c - 16N_f)\rho} \\
&\quad - \frac{(N_c - N_f)N_f(2N_c - N_f + 2Q_o)}{32(-2N_c + N_f)^2\rho^2} + \mathcal{O}(\rho^{-3}), \\
Y &= \frac{1}{4}(-N_c + N_f) + \frac{N_f(-N_c + N_f)}{(64N_c - 32N_f)\rho^2} + \frac{(N_c - N_f)N_f(2N_c - N_f + 2Q_o)}{32(-2N_c + N_f)^2\rho^3} \\
&\quad + \mathcal{O}(\rho^{-4}), \\
e^{4(\phi - \phi_o)} &= e^{4(\rho - \rho_o)} \left( -\frac{1}{2((N_c - N_f)^2(2N_c - N_f))\rho} + \frac{2N_c - 3N_f + 4Q_o}{8(2N_c^2 - 3N_cN_f + N_f^2)^2\rho^2} \right. \\
&\quad \left. + \mathcal{O}(\rho^{-3}) \right) \sinh^2(2\rho_o), \tag{4.12} \\
a &= e^{-2(\rho - \rho_o)} \left( 1 + \frac{N_c - N_f}{(4N_c - 2N_f)\rho} - \frac{(N_c - N_f)(2N_c - N_f + 4Q_o)}{8(-2N_c + N_f)^2\rho^2} + \mathcal{O}(\rho^{-2}) \right).
\end{aligned}$$

Clearly, these expansions reproduce in a unified way both the type **A** UV expansions of eq. (3.6) in [12] and the type **N** UV expansions of eqs. (4.20) in [9].

(iii)  $N_f = 2N_c$ :

Finally, for  $N_f = 2N_c$  the type **A** class I UV solution is given by the exact solution (4.5), while the type **N** class I solution is again given by (4.5) up to exponentially suppressed terms. Namely

$$P = N_c + \sqrt{N_c^2 + Q_o^2} + \mathcal{O}(e^{-4(\rho - \rho_o)}) = \frac{8N_c}{(4 - \xi)\xi} + \mathcal{O}(e^{-4(\rho - \rho_o)}), \quad 0 < \xi < 4. \tag{4.13}$$

Using (3.16)-(3.19) this leads to the expansions

$$\begin{aligned}
e^{2h} &= \frac{N_c}{\xi} + \mathcal{O}(e^{-4(\rho-\rho_o)}), \\
e^{2g} &= \frac{4N_c}{4-\xi} + \mathcal{O}(e^{-4(\rho-\rho_o)}), \\
Y &= \frac{N_c}{4} + \mathcal{O}(e^{-4(\rho-\rho_o)}), \\
e^{4(\phi-\phi_o)} &= e^{4(\rho-\rho_o)} \sinh^2(2\rho_o) \frac{(4-\xi)\xi}{16N_c^3} (1 + \mathcal{O}(e^{-4(\rho-\rho_o)})), \\
a &= \frac{4}{\xi} e^{-2(\rho-\rho_o)} + \mathcal{O}(e^{-4(\rho-\rho_o)}).
\end{aligned} \tag{4.14}$$

Note that as  $\rho_o \rightarrow -\infty$ , this reproduces the exact solution (4.5).

These asymptotic expansions exhaust all possible UV behaviors of the physically accepted solutions of eq. (3.20), with  $P$  growing at most linearly with  $\rho$  as  $\rho \rightarrow \infty$ . As we have already mentioned, these expansions receive exponentially suppressed corrections, which are of two different types. Firstly, if  $\rho_o > -\infty$ , i.e. we are interested in the type **N** solutions, then, as we have indicated, there are exponentially suppressed corrections due to the presence of  $\cosh \tau$  in  $Q$  and in eq. (3.20). Moreover, the expansions of eqs. (4.9), (4.11) and (4.13) do not involve any integration constants (except for  $Q_o$  and  $\rho_o$  which appear as parameters in (3.20)). As we shall see below, turning on one of the two integration constants of (3.20) will change the UV asymptotics to that of class II. Turning on the second integration constant, however, will introduce certain exponentially suppressed terms. See e.g. [12] for a discussion of these terms.

### **Class II:**

The second class of UV asymptotics consists of solutions where  $P \sim c_+ e^{4\rho/3}$  as  $\rho \rightarrow \infty$ , independently of the values of the parameters  $N_c$ ,  $N_f$ ,  $Q_o$  or  $\rho_o$ . The positive constant  $c_+$  here corresponds to one of the integration constants of (3.20), and it completely determines the leading asymptotic behavior of the solution. Keeping the leading exponentially suppressed corrections that differentiate the expansions of the type **A** and the type **N** solutions in this

case, we have respectively

$$\begin{aligned}
P = & e^{4\rho/3} \left\{ c_+ + \frac{9N_f}{8} e^{-4\rho/3} + \frac{1}{64c_+} [64(2N_c - N_f)^2 \rho^2 + 128(2N_c - N_f)Q_o \rho + \right. \\
& 9(4N_c - 3N_f)(4N_c - N_f) + 64Q_o^2] e^{-8\rho/3} \\
& - \frac{1}{64c_+^2} \left[ \frac{64}{3} N_f(2N_c - N_f)^2 \rho^3 + 16N_f(2N_c - N_f)(3(2N_c - N_f) + 4Q_o) \rho^2 \right. \\
& \left. \left. + (3N_f(32N_c(N_c - N_f) + 5N_f^2) + 32N_f Q_o(2Q_o - 3N_f + 6N_c)) \rho + c_- \right] e^{-12\rho/3} \right. \\
& \left. + \mathcal{O}(\rho^3 e^{-16\rho/3}) \right\}, \tag{4.15}
\end{aligned}$$

for  $\rho_o \rightarrow -\infty$  (type **A** solution), and

$$\begin{aligned}
P = & e^{4\rho/3} \left\{ c_+ \left( 1 - \frac{8}{3} \rho e^{-4\rho} + \mathcal{O}(e^{-8\rho}) \right) + \frac{9N_f}{8} (1 + \mathcal{O}(\rho e^{-4\rho})) e^{-4\rho/3} \right. \\
& + \frac{1}{64c_+} [64(2N_c - N_f)^2 \rho^2 + 128(2N_c - N_f)Q_o \rho + 144N_c(N_c - N_f) + 27N_f^2 \\
& + 64Q_o^2] e^{-8\rho/3} \\
& - \frac{1}{64c_+^2} \left[ \frac{64}{3} N_f(2N_c - N_f)^2 \rho^3 + 16N_f(2N_c - N_f)(3(2N_c - N_f) + 4Q_o) \rho^2 \right. \\
& \left. \left. + (3N_f(32N_c(N_c - N_f) + 5N_f^2) + 32bQ_o(2Q_o - 3N_f + 6N_c)) \rho + c_- \right] e^{-12\rho/3} \right. \\
& \left. + \mathcal{O}(\rho^3 e^{-16\rho/3}) \right\}, \tag{4.16}
\end{aligned}$$

for  $\rho_o > -\infty$  (type **N** solution), where we have set  $\rho_o = 0$  for convenience. Notice that these two expansions involve two integration constants,  $c_+ > 0$  and  $c_-$ , and are identical except for the exponentially suppressed corrections in the first line of (4.16), coming from the expansion of  $\cosh \tau$ . Using eqs. (3.16)-(3.19) these expansions give to first subleading order

$$\begin{aligned}
e^{2h} &= \frac{1}{4} \left( c_+ e^{4\rho/3} + (2N_c - N_f) \rho + Q_o + \frac{9N_f}{8} + \mathcal{O}(\rho^2 e^{-4\rho/3}) \right), \\
e^{2g} &= c_+ e^{4\rho/3} - (2N_c - N_f) \rho - Q_o + \frac{9N_f}{8} + \mathcal{O}(\rho^2 e^{-4\rho/3}), \\
Y &= \frac{1}{8} \left( \frac{4c_+}{3} e^{4\rho/3} + N_f + \mathcal{O}(\rho^2 e^{-4\rho/3}) \right), \\
e^{4(\phi - \phi_o)} &= 1 - \frac{3N_f}{c_+} e^{-4\rho/3} + \mathcal{O}(\rho^2 e^{-8\rho/3}), \\
a &= 2e^{-2(\rho - \rho_o)} \left( 1 + \frac{1}{c_+} ((2N_c - N_f) \rho + Q_o) e^{-4\rho/3} + \mathcal{O}(\rho^2 e^{-8\rho/3}) \right). \tag{4.17}
\end{aligned}$$

As it can be seen directly by inserting these asymptotics in the metric (2.13), the geometry asymptotes to the conifold. These solutions then correspond to turning on an irrelevant operator with coupling  $c_+$  that drives us away from the near horizon geometry of the D5 branes (cf. [19, 20]). We will give arguments supporting this interpretation in Section 5.

In fact, it turns out that the geometry asymptotes to the undeformed conifold for the type **A** case, while it asymptotes to the deformed conifold in the type **N** case. In order to see this, however, we need to resum certain terms in the asymptotic expansion in eqs. (4.16). This can be done as follows. Since the powers of  $e^{4\rho/3}$  in these expansions for large  $\rho$  come with powers of  $c_+$ , one can, instead of asymptotic solutions for large  $\rho$ , look for solutions of (3.20) in an expansion for large  $c_+$ . In Appendix B we construct systematically both the type **A** and type **N** solutions in the large  $c_+$  expansion, showing that the former asymptote to the conifold, while the latter to the deformed conifold.

### 4.3.2 IR asymptotics

Let us next analyze the IR behavior of the solutions of (3.20). By infrared we generically mean the minimum value,  $\rho_{IR}$ , of the radial coordinate, where the geometry ends. For the type **N** backgrounds  $\rho \geq \rho_o > -\infty$  and so necessarily  $\rho_{IR} \geq \rho_o > -\infty$ . For the type **A** backgrounds, however,  $\rho_o \rightarrow -\infty$  and so  $\rho_{IR}$  is totally unconstrained. In contrast to the UV asymptotics where we always have  $\rho \rightarrow \infty$  (unless we consider Landau poles), the first task in determining the IR asymptotics is to determine the location of  $\rho_{IR}$ . This is not too complicated, though, since there are two possible inequivalent locations for  $\rho_{IR}$  in each case. Namely, for the type **A** backgrounds ( $\tau = 0$ ) it is either located at  $\rho_{IR} = -\infty$  or at a finite radial distance  $\rho_{IR} > -\infty$ , which can be taken to be zero by a shift in the radial coordinate. For the type **N** case the IR is located either at  $\rho_{IR} = \rho_o$  ( $\tau(\rho_{IR}) = \infty$ ) or at  $\rho_{IR} > \rho_o$  ( $\tau(\rho_{IR}) < \infty$ ), in which case again it can be taken to be zero by a shift of the radial coordinate. In summary, then

$$\rho_{IR} = \begin{cases} 0 & \text{or } -\infty, & \mathbf{A}, \\ 0 & \text{or } \rho_o, & \mathbf{N}. \end{cases} \quad (4.18)$$

Looking at the metric (3.1) and the dilaton (3.18) we see that the IR will generically be located where some of the functions  $Y$ ,  $H = (P + Q)/4$ ,  $G = (P - Q)/4$  become zero<sup>6</sup>. Since the dilaton must remain finite for solutions with acceptable “good” singularities, this means that at the same time some of these functions, or  $\sinh \tau$ , must go to infinity sufficiently fast.

Another possibility for the location of the IR is that none of the functions  $H$ ,  $G$ ,  $Y$  vanish, but instead some of them, or  $\sinh \tau$ , go to infinity, forcing  $e^\phi \rightarrow 0$ . However, if either  $H$  or  $G$

---

<sup>6</sup>Note that from (3.1) we see that the geometry degenerates when  $P \cosh \tau \pm Q = 0$ , while from (3.18) we see that the dilaton will generically diverge if  $P \pm Q = 0$  (unless  $Y$  or  $\sinh \tau$  go to infinity suitably fast). For the type **A** case these two conditions coincide, but for the type **N** case  $P \pm Q$  go to zero necessarily before  $P \cosh \tau \pm Q$ . Hence, in either case, one should use the condition  $P \pm Q = 0$  as a criterion for the location of the IR.

goes to infinity, then  $P \rightarrow \infty$  in the IR. But given that  $P'$  should remain finite due to (3.19), it is easy to show that equation (3.20) excludes this possibility. Therefore,  $H$  and  $G$  must remain finite, while  $Y$  and/or  $\sinh \tau$  would have to go to infinity. An easy calculation using the fact that  $P' \sim Y$  and eq. (3.20) excludes the possibility  $Y \rightarrow \infty$  while  $P$  remains finite as  $\rho \rightarrow \rho_{IR}$ . The only possibility of this kind then would be that all  $H$ ,  $G$  and  $Y$  remain finite and non-zero, while  $\sinh \tau \rightarrow \infty$ . This possibility is also easily excluded by eq. (3.20).

We therefore conclude that all possible IRs are classified by the vanishing of some of the functions  $H$ ,  $G$  and  $Y$ . One expects that depending on which of these functions vanish, there will be qualitatively different IR behaviors. Following [12] we will call type I the case where  $Y \rightarrow 0$ , type II the case where *either*  $H$  or  $G$  vanish, and type III the case where *both*  $H$  and  $G$  vanish. In the type **A** case and only for  $N_c = 0$  or  $N_f = N_c$  there is one more case where respectively  $G$  or  $H$  vanishes in addition to  $Y$ , but we will include this in the type I case. As we shall see, the type I backgrounds have  $\rho_{IR} = -\infty$  in the type **A** case and  $\rho_{IR} = \rho_o = 0$  in the type **N**, whereas the type II and type III backgrounds have  $\rho_{IR} = 0$  in both cases.

### Type I:

By definition, these are solutions that have  $Y \rightarrow 0$  as  $\rho \rightarrow \rho_{IR}$ . Now there is an exact solution of eq. (3.20), valid for arbitrary parameters  $N_c$ ,  $N_f$ ,  $Q_o$  and  $\rho_o$ , namely

$$P = -N_f \rho + P_o, \quad \rho \leq P_o/N_f, \quad (4.19)$$

where  $P_o$  is an arbitrary constant. This solution, via (3.19), leads to  $Y = 0$  identically, and it is therefore not an acceptable solution. Since it involves only one integration constant,  $P_o$ , though, there is one more integration constant left which can be used to deform this solution to a non-singular solution. Together these two integration constants parametrize *any* solution of (3.20) in the vicinity of  $Y = 0$ . However, since in the solution (4.19)  $\rho$  is bounded from above, the deformed solution must deviate strongly from (4.19) in order to match with the UV asymptotics described above. One can then construct the type I IR expansions by solving (3.20) perturbatively in the second integration constant around the unphysical solution (4.19). In Appendix B we systematically construct these expansions both in the type **A** and type **N** cases. Since (4.19) solves eq. (B.3) with  $c_+ = 0$ , these expansions are expansions in powers of  $c_+^3$ . From these then we obtain the following type I IR expansions.

For the type **A** backgrounds we find that  $\rho_{IR} \rightarrow -\infty$  and

$$\begin{aligned} P = & -N_f \rho + P_o + c_+^3 e^{4\rho} \left( N_c(N_f - N_c) \rho^2 - \frac{1}{2}(N_c(N_f - N_c) + N_f P_o + (2N_c - N_f)Q_o) \rho \right. \\ & \left. + \frac{1}{4}(P_o^2 - Q_o^2) + \frac{1}{8}(N_f P_o + (2N_c - N_f)Q_o + N_c(N_f - N_c)) \right) + \mathcal{O}(c_+^6 e^{8\rho} \rho^3). \end{aligned} \quad (4.20)$$

Evaluating the rest of the functions parametrizing the background using (3.16)-(3.19), we find

$$\begin{aligned}
H &= \frac{1}{2}(N_c - N_f)\rho + \frac{1}{4}(P_o + Q_o) \\
&\quad + c_+^3 e^{4\rho} \left( \frac{1}{4}N_c(-N_c + N_f)\rho^2 + \frac{1}{8}(N_c^2 + N_f(-P_o + Q_o) - N_c(N_f + 2Q_o))\rho \right. \\
&\quad \left. + \frac{1}{32}(-N_c^2 + N_c(N_f + 2Q_o) + (P_o - Q_o)(N_f + 2(P_o + Q_o))) \right) + \mathcal{O}(c_+^6 e^{8\rho} \rho^3), \\
G &= -\frac{N_c \rho}{2} + \frac{1}{4}(P_o - Q_o) \\
&\quad + c_+^3 e^{4\rho} \left( \frac{1}{4}N_c(-N_c + N_f)\rho^2 + \frac{1}{8}(N_c^2 + N_f(-P_o + Q_o) - N_c(N_f + 2Q_o))\rho \right. \\
&\quad \left. + \frac{1}{32}(-N_c^2 + N_c(N_f + 2Q_o) + (P_o - Q_o)(N_f + 2(P_o + Q_o))) \right) + \mathcal{O}(c_+^6 e^{8\rho} \rho^3), \\
Y &= c_+^3 e^{4\rho} \left( \frac{1}{2}N_c(-N_c + N_f)\rho^2 + \frac{1}{4}(-2N_c Q_o + N_f(-P_o + Q_o))\rho + \frac{1}{8}(P_o^2 - Q_o^2) \right) \\
&\quad + \mathcal{O}(c_+^6 e^{8\rho} \rho^3), \\
e^{4(\phi - \phi_o)} &= \frac{8}{c_+^3 (4N_c(-N_c + N_f)\rho^2 + 2(-2N_c Q_o + N_f(-P_o + Q_o))\rho + (P_o^2 - Q_o^2)) + \mathcal{O}(e^{4\rho})}.
\end{aligned} \tag{4.21}$$

This asymptotic solution then coincides with the type I IR solution of [12] (cf. eqs. (3.7)-(3.9)). However, when either  $N_f = N_c$  and  $P_o = -Q_o$  or  $N_c = 0$  and  $P_o = Q_o$ , we find a second asymptotic solution, namely

$$P = -N_f \rho + P_o + c_+^3 e^{2\rho} \sqrt{-N_f \rho + P_o + N_f/4} + \dots, \tag{4.22}$$

which gives

$$\begin{aligned}
H &= \frac{1}{2}(N_c - N_f)\rho + \frac{1}{4}(P_o + Q_o) + \frac{1}{4}c_+^3 e^{2\rho} \sqrt{-N_f \rho} + \mathcal{O}(e^{2\rho} |\rho|^{-1/2}), \\
G &= -\frac{N_c \rho}{2} + \frac{1}{4}(P_o - Q_o) + \frac{1}{4}c_+^3 e^{2\rho} \sqrt{-N_f \rho} + \mathcal{O}(e^{2\rho} |\rho|^{-1/2}), \\
Y &= \frac{1}{4}c_+^3 e^{2\rho} \sqrt{-N_f \rho} + \mathcal{O}(e^{2\rho} |\rho|^{-1/2}), \\
e^{4(\phi - \phi_o)} &= \frac{4(-N_f \rho)^{-1/2} e^{2\rho}}{c_+^3 (4N_c(-N_c + N_f)\rho^2 + 2(-2N_c Q_o + N_f(-P_o + Q_o))\rho + (P_o^2 - Q_o^2))} + \dots
\end{aligned} \tag{4.23}$$

For the type **N** backgrounds, we find that  $\rho_{IR} = \rho_o$ , which we can take to be zero without loss of generality. Moreover we must set  $Q_o = -(2N_c - N_f)/2$  in order for a well defined

solution of this form to exist. The asymptotic solution then takes the form

$$P = -N_f \rho + P_o + \frac{4}{3} c_+^3 P_o^2 \rho^3 - 2c_+^3 N_f P_o \rho^4 + \frac{4}{5} c_+^3 \left( \frac{4}{3} P_o^2 + N_f^2 \right) \rho^5 + \mathcal{O}(\rho^6), \quad (4.24)$$

which leads to

$$\begin{aligned} e^{2h} &= \frac{P_o \rho}{2} - \frac{N_f \rho^2}{2} - \frac{2P_o \rho^3}{3} + \mathcal{O}(\rho^4), \\ e^{2g} &= \frac{P_o}{2\rho} - \frac{N_f}{2} + \frac{2P_o \rho}{3} + \frac{2}{3} (-2N_c + c_+^3 P_o^2) \rho^2 - \frac{1}{45} ((8 + 45c_+^3 N_f) P_o) \rho^3 + \mathcal{O}(\rho^4), \\ Y &= \frac{1}{2} c_+^3 P_o^2 \rho^2 - c_+^3 N_f P_o \rho^3 + \frac{1}{6} c_+^3 (3N_f^2 + 4P_o^2) \rho^4 + \mathcal{O}(\rho^5), \\ e^{4(\phi - \phi_o)} &= 1 + \frac{4N_f \rho}{P_o} + \frac{10N_f^2 \rho^2}{P_o^2} + \left( \frac{20N_f^3}{P_o^3} - \frac{8N_f}{3P_o} - \frac{8c_+^3 P_o}{3} \right) \rho^3 + \mathcal{O}(\rho^4), \\ a &= 1 - 2\rho^2 - \frac{4(-2N_c + N_f) \rho^3}{3P_o} + \frac{2(4N_c N_f - 2N_f^2 + 5P_o^2) \rho^4}{3P_o^2} + \mathcal{O}(\rho^5). \end{aligned} \quad (4.25)$$

Identifying then  $2c_+^3 P_o^2 = c_2 N_c$  and  $c_1 = 4(2N_c - N_f)/3P_o$  the expansion (4.25) exactly reproduces the expansion of eq. (4.21) in [9]. Finally, note there is another isolated type I solution, namely the exact solution in eq. (4.1).

## Type II:

Type II infrared behavior corresponds to  $H = 0$  or  $G = 0$ . Let us first assume that this behavior occurs when the IR is located at  $\rho_{IR} > \rho_o$ . Without loss of generality we can choose  $\rho_{IR} = 0$ . With this choice we then necessarily have  $\rho_o < 0$ . Expanding  $Q$  in (3.17) around  $\rho = 0$  we obtain

$$Q = b_0 + b_1 \rho + \mathcal{O}(\rho^2), \quad (4.26)$$

where

$$\begin{aligned} b_0 &= -\coth(2\rho_o) \left( Q_o + \frac{2N_c - N_f}{2} \right) - \frac{2N_c - N_f}{2}, \\ b_1 &= -\frac{2}{\sinh^2(2\rho_o)} \left( Q_o + \frac{2N_c - N_f}{2} \right) - (2N_c - N_f) \coth(2\rho_o), \\ b_2 &= \dots \end{aligned} \quad (4.27)$$

Looking for IR solutions of (3.20) with  $G \rightarrow 0$  as  $\rho \rightarrow 0$  we find we must require that  $b_0 > 0$ . The corresponding asymptotic solution then takes the form

$$\begin{aligned} P &= Q + h_1 \rho^{1/2} - \frac{1}{6b_0} (h_1^2 + 12b_0(b_1 + N_f)) \rho \\ &\quad + \frac{h_1}{72b_0^2} (5h_1^2 + 6(5b_1 + 2N_f)b_0 - 72b_0^2 \coth(2\rho_o)) \rho^{3/2} + \mathcal{O}(\rho^2), \end{aligned} \quad (4.28)$$



where  $h_1$  is an arbitrary constant. Note that this expansion for  $P$  admits a smooth limit when  $\rho_o \rightarrow -\infty$  and so it is valid for both type **A** ( $\rho_o \rightarrow -\infty$ ) and type **N** backgrounds. Using then (3.16)-(3.19) we get

$$\begin{aligned}
H &= \frac{b_0}{2} + \frac{h_1 \rho^{1/2}}{4} + \left( -\frac{h_1^2}{24b_0} - \frac{N_f}{2} \right) \rho \\
&\quad + \frac{h_1 (30b_1 b_0 - 72 \coth(2\rho_o) b_0^2 + 5h_1^2 + 12b_0 N_f) \rho^{3/2}}{288b_0^2} + \mathcal{O}(\rho^2), \\
G &= \frac{h_1 \rho^{1/2}}{4} + \left( -\frac{b_1}{2} - \frac{h_1^2}{24b_0} - \frac{N_f}{2} \right) \rho \\
&\quad + \frac{h_1 (30b_1 b_0 - 72 \coth(2\rho_o) b_0^2 + 5h_1^2 + 12b_0 N_f) \rho^{3/2}}{288b_0^2} + \mathcal{O}(\rho^2), \\
Y &= \frac{h_1}{16\rho^{1/2}} + \frac{1}{48} \left( -6b_1 - \frac{h_1^2}{b_0} - 6N_f \right) \\
&\quad + \frac{h_1 (30b_1 b_0 - 72 \coth(2\rho_o) b_0^2 + 5h_1^2 + 12b_0 N_f) \rho^{1/2}}{384b_0^2} + \mathcal{O}(\rho), \\
e^{4(\phi - \phi_o)} &= 1 + \frac{4(b_1 + N_f) \rho^{1/2}}{h_1} \\
&\quad + \frac{2(18b_1^2 b_0 - b_1(h_1^2 - 36b_0 N_f) + 2N_f(h_1^2 + 9b_0 N_f)) \rho}{3b_0 h_1^2} + \mathcal{O}(\rho^{3/2}), \\
a &= \frac{\sinh^{-1}(2\rho_o)}{1 + \coth(2\rho_o)} + \frac{\sinh^{-1}(2\rho_o) h_1 \rho^{1/2}}{(1 + \coth(2\rho_o))^2 b_0} + \mathcal{O}(\rho). \tag{4.29}
\end{aligned}$$

Note that we have given the functions  $H$  and  $G$  here instead of the old variables  $e^{2h}$  and  $e^{2g}$ . The reason is that the expansion of these functions takes the form

$$\begin{aligned}
e^{2h} &= -\frac{h_1 \rho^{1/2}}{2 + \coth(2\rho_o) + \tanh(2\rho_o) + \mathcal{O}(\rho^{1/2})} + \mathcal{O}(\rho), \\
e^{2g} &= -(1 + \coth(2\rho_o)) b_0 - \coth(2\rho_o) h_1 \rho^{1/2} + \mathcal{O}(\rho), \tag{4.30}
\end{aligned}$$

and so it is clear that the expansion around  $\rho \rightarrow 0$  of  $e^{2h}$  does not commute with the  $\rho_o \rightarrow -\infty$  (type **A**) limit. In particular, to evaluate this limit one needs to keep the subleading  $\mathcal{O}(\rho^{1/2})$  term in the denominator of  $e^{2h}$ . The same is true for  $a$ , whose expansion, keeping the subleading term in the denominator, takes the form

$$a = \frac{b_0}{b_0 e^{2\rho_o} + \cosh(2\rho_o) h_1 \rho^{1/2} + \dots} + \dots \tag{4.31}$$

In the form we have given the asymptotic solution, however, one can directly take the limit  $\rho_o \rightarrow -\infty$ , except in  $a$  which vanishes in this limit once one takes into account the sub-

leading corrections. In this limit this asymptotic solution reduces to the asymptotic solution in eq. (3.11) of [12].<sup>7</sup> The expansion here gives the generalization to the type **N** backgrounds.

Similarly, looking for IR solutions of (3.20) with  $H \rightarrow 0$  as  $\rho \rightarrow 0$  we find we must require that  $b_0 < 0$  and then,

$$\begin{aligned} P = & -Q + h_1 \rho^{1/2} + \frac{1}{6b_0} (h_1^2 + 12b_0(b_1 - N_f)) \rho \\ & + \frac{h_1}{72b_0^2} (5h_1^2 + 6(5b_1 - 2N_f)b_0 - 72b_0^2 \coth(2\rho_o)) \rho^{3/2} + \mathcal{O}(\rho^2), \end{aligned} \quad (4.32)$$

where  $h_1$  is an arbitrary constant. From (3.16)-(3.19) then we get

$$\begin{aligned} e^{2h} &= -\frac{h_1 \rho^{1/2}}{-2 + \coth(2\rho_o) + \tanh(2\rho_o)} \\ &\quad - \frac{(6(-1 + \coth(2\rho_o))b_o(b_1 - N_f) + h_1^2(1 + \coth(2\rho_o) + \tanh(2\rho_o))) \rho}{6b_o(-1 + \coth(2\rho_o))^2} + \mathcal{O}(\rho^{3/2}), \\ e^{2g} &= (-1 + \coth(2\rho_o))b_o - \coth(2\rho_o)h_1 \rho^{1/2} \\ &\quad + \left( \coth(2\rho_o) \left( 2N_f - b_1 - \frac{h_1^2}{6b_o} \right) + 2b_o \sinh^{-2}(2\rho_o) - b_1 \right) \rho + \mathcal{O}(\rho^{3/2}), \\ Y &= \frac{h_1}{16\rho^{1/2}} + \frac{1}{48} \left( 6b_1 + \frac{h_1^2}{b_o} - 6N_f \right) \\ &\quad + \frac{h_1(30b_1b_o - 72\coth(2\rho_o)b_o^2 + 5h_1^2 - 12b_oN_f)\rho^{1/2}}{384b_o^2} + \mathcal{O}(\rho), \\ e^{4(\phi - \phi_o)} &= 1 - \frac{4(b_1 - N_f)\rho^{1/2}}{h_1} \\ &\quad + \frac{2(18b_1^2b_o + 2N_f(-h_1^2 + 9b_oN_f) - b_1(h_1^2 + 36b_oN_f))\rho}{3(b_o h_1^2)} + \mathcal{O}(\rho^{3/2}), \\ a &= \frac{\sinh^{-1}(2\rho_o)}{-1 + \coth(2\rho_o)} + \frac{\sinh^{-1}(2\rho_o)h_1 \rho^{1/2}}{(-1 + \coth(2\rho_o))^2 b_o} + \mathcal{O}(\rho). \end{aligned} \quad (4.33)$$

In the limit  $\rho_o \rightarrow -\infty$  (type **A**) these expansions reproduce the expansions of eq. (3.10) in [12]. Here we have generalized this IR solution to the type **N** backgrounds.

Let us finally consider the case  $\rho_{IR} = \rho_o$ . Note that in this case  $Q$  has a pole at  $\rho_o$  unless  $Q_o = -(2N_c - N_f)(\rho_o + \frac{1}{2})$ . Setting then, without loss of generality,  $\rho_o = 0$ , we have

$$Q = (2N_c - N_f) \left( \frac{2}{3}\rho^2 - \frac{8}{45}\rho^4 + \frac{64}{945}\rho^6 + \mathcal{O}(\rho^8) \right). \quad (4.34)$$

Since we are looking for solutions such that  $H$  or  $G$  go to zero as  $\rho \rightarrow 0$ , it follows that in this case *both*  $H$  and  $G$  will go to zero as  $\rho \rightarrow 0$  and hence this will be a type III solution, which we will consider below.

---

<sup>7</sup>Note that  $h_1$  here differs by a factor of 4 from  $h_1$  in [12].

### Type III:

Finally we look for IR solutions for which  $H \rightarrow 0$  and  $G \rightarrow 0$  in the IR. We again first consider  $\rho_{IR} > \rho_o$  and we take  $\rho_{IR} = 0$ . In terms of the expansion (4.26) this requires that  $b_0 = 0$ . We then find,

$$P = h_1 \rho^{1/3} - \frac{9N_f}{5} \rho - \frac{2h_1}{3} \coth(2\rho_o) \rho^{4/3} - \frac{1}{175h_1} (50b_1^2 - 18N_f^2) \rho^{5/3} + \mathcal{O}(\rho^2), \quad (4.35)$$

where  $h_1 \neq 0$  is an arbitrary constant. Evaluating the rest of the background functions we find

$$\begin{aligned} e^{2h} &= -\frac{1}{4} h_1 \tanh(2\rho_o) \rho^{1/3} + \frac{1}{20} \tanh(2\rho_o) (9N_f + 5b_1 \tanh(2\rho_o)) \rho \\ &\quad + \frac{1}{6} \left( 1 + \frac{3}{\cosh^2(2\rho_o)} \right) h_1 \rho^{4/3} + \mathcal{O}(\rho^{5/3}), \\ e^{2g} &= -\coth(2\rho_o) h_1 \rho^{1/3} + \left( -b_1 + \frac{9}{5} \coth(2\rho_o) N_f \right) \rho \\ &\quad + \frac{1}{3} (-5 + \cosh(2\rho_o)) \sinh^{-2}(2\rho_o) h_1 \rho^{4/3} + \mathcal{O}(\rho^{5/3}), \\ Y &= \frac{h_1}{24\rho^{2/3}} - \frac{N_f}{10} - \frac{1}{9} \coth(2\rho_o) h_1 \rho^{1/3} + \frac{(-25b_1^2 + 9N_f^2) \rho^{2/3}}{420h_1} + \mathcal{O}(\rho), \\ e^{4(\phi-\phi_o)} &= 1 + \frac{6N_f \rho^{2/3}}{h_1} + \frac{3(5b_1^2 + 39N_f^2) \rho^{4/3}}{5h_1^2} + \mathcal{O}(\rho^{5/3}), \\ a &= \frac{1}{\cosh(2\rho_o)} - \frac{b_1 \tanh^2(2\rho_o) \rho^{2/3}}{h_1 \sinh(2\rho_o)} + \frac{2 \tanh^2(2\rho_o)}{\sinh(2\rho_o)} \rho + \mathcal{O}(\rho^{4/3}). \end{aligned} \quad (4.36)$$

This expansion generalizes the type **A** III expansion of eq. (3.12) in [12] to the type **N** backgrounds. Indeed, eq. (4.35) reproduces this expansion in the type **A** limit  $\rho_o \rightarrow -\infty$ .

Coming finally back to the case  $\rho_{IR} = \rho_o$ , we have seen that one should set  $Q_o = -(2N_c - N_f)(\rho_o + \frac{1}{2})$  to avoid the pole in  $Q$ , which takes the form  $(\rho_o = 0)$  (4.34). Looking for solutions with  $P \rightarrow 0$  as  $\rho \rightarrow 0$ , and noting that  $P$  should go to zero slower than  $Q$  to ensure that both  $H$  and  $G$  remain positive, we find that there is a solution provided  $N_f = 0$ . Namely,

$$P = h_1 \rho + \frac{4h_1}{15} \left( 1 - \frac{4N_c^2}{h_1^2} \right) \rho^3 + \frac{16h_1}{525} \left( 1 - \frac{4N_c^2}{3h_1^2} - \frac{32N_c^4}{3h_1^4} \right) \rho^5 + \mathcal{O}(\rho^7), \quad (4.37)$$

where  $h_1$  is again an arbitrary constant. This gives,

$$\begin{aligned}
e^{2h} &= \frac{h_1 \rho^2}{2} + \frac{4}{45} \left( -6h_1 + 15N_c - \frac{16N_c^2}{h_1} \right) \rho^4 + \mathcal{O}(\rho^6), \\
e^{2g} &= \frac{h_1}{2} + \frac{4}{15} \left( 3h_1 - 5N_c - \frac{2N_c^2}{h_1} \right) \rho^2 + \frac{8(3h_1^4 + 70h_1^3 N_c - 144h_1^2 N_c^2 - 32N_c^4) \rho^4}{1575h_1^3} \\
&\quad + \mathcal{O}(\rho^6), \\
Y &= \frac{h_1}{8} + \frac{(h_1^2 - 4N_c^2) \rho^2}{10h_1} + \frac{(6h_1^4 - 8h_1^2 N_c^2 - 64N_c^4) \rho^4}{315h_1^3} + \mathcal{O}(\rho^6), \\
e^{4(\phi - \phi_o)} &= 1 + \frac{64N_c^2 \rho^2}{9h_1^2} + \frac{128N_c^2 (-15h_1^2 + 124N_c^2) \rho^4}{405h_1^4} + \mathcal{O}(\rho^6), \\
a &= 1 + \left( -2 + \frac{8N_c}{3h_1} \right) \rho^2 + \frac{2(75h_1^3 - 232h_1^2 N_c + 160h_1 N_c^2 + 64N_c^3) \rho^4}{45h_1^3} + \mathcal{O}(\rho^6).
\end{aligned} \tag{4.38}$$

This is a new IR solution with a ‘good singularity’ for the type **N** backgrounds in the *flavorless* case. It should be interesting to study its dynamics.

#### 4.4 Solutions of the BPS equations as RG flows

The solutions described in [9, 12] are a special case of the set of possible solutions to the BPS equations, given the ansatz in eq. (3.1). We have seen that more general asymptotics are possible, for large values of the radial coordinate  $\rho$ , the metric elements can grow as  $H \sim G \sim Y \sim e^{4\rho/3}$ . Another possibility is that one of the elements of the metric can collapse at a finite value of the radial coordinate, ending the space. In order to study how generic these solutions are; we can make a qualitative analysis of the system of BPS equations as a four-dimensional dynamical system. In the type **A** case the system is simpler and reduces to three dimensions. We would like to interpret each solution as describing a different RG flow in the space of holographic dual theories.

We will work with the  $\rho$ -dependent functions  $P = N_c p$ ,  $Q = N_c q$  and  $u = (p^2 - q^2)Y/N_c$  and use the parameter  $x \equiv N_f/N_c$ . Using the first order equations in (3.12), with the constraint

(3.15) imposed, we can rewrite the BPS equations as,

$$\begin{aligned}
p' &= -x + \frac{8u}{p^2 - q^2}, \\
q' &= \frac{2-x}{\cosh \tau} - 2q \sinh \tau \tanh \tau, \\
u' &= 4u \left[ \cosh \tau - \frac{xp}{p^2 - q^2} - \frac{q}{p^2 - q^2} \left( \frac{2-x}{\cosh \tau} - 2q \sinh \tau \tanh \tau \right) \right], \\
\tau' &= -2 \sinh \tau, \\
\phi' &= \frac{xp}{p^2 - q^2} + \frac{q}{p^2 - q^2} \left( \frac{2-x}{\cosh \tau} - 2q \sinh \tau \tanh \tau \right).
\end{aligned} \tag{4.39}$$

Let us fix  $1 < x \leq 2$ ,<sup>8</sup> since the equations are independent of  $\phi$ , the solutions can be described as trajectories in the  $(p, q, u, \tau)$  space. The trajectories are tangent to the vector field defined by the values of  $(p', q', u', \tau')$ . The BPS type **A** equations correspond to  $\tau = 0$ . That is a co-dimension one fixed subspace in the  $\tau$  direction.

For physical solutions all the functions (appearing as warp factors in the metric) should be positive, so  $p^2 \geq q^2$ . When  $\tau > 0$ ,  $\tau' < 0$ , so type **N** solutions flow from a finite or infinite value of  $\tau$  in the IR ( $\rho = \rho_{IR}$ ) to  $\tau = 0$  in the UV ( $\rho \rightarrow \infty$ ). This is another way to see that types **A** and **N** solutions have the same UV asymptotics. At fixed values of  $\tau > 0$  and  $u$  there are three relevant curves in the  $(p, q)$  plane

$$\begin{aligned}
p' = 0 &\Rightarrow p^2 - q^2 = \frac{8u}{x}, \quad \text{if } p^2 - q^2 > \frac{8u}{x}, \quad \text{then } p' < 0, \\
q' = 0 &\Rightarrow q = \frac{2-x}{(2 \sinh^2 \tau)}, \quad \text{if } q > \frac{2-x}{(2 \sinh^2 \tau)}, \quad \text{then } q' < 0,
\end{aligned} \tag{4.40}$$

and the curve  $u' = 0$

$$\left( p - \frac{x}{2 \cosh \tau} \right)^2 - (1 - 2 \tanh^2 \tau) \left( q + \frac{2-x}{2(1 - \sinh^2 \tau)} \right)^2 = \frac{x^2}{4 \cosh^2 \tau} - \frac{(2-x)^2}{4 \cosh^2 \tau (1 - \sinh^2 \tau)}. \tag{4.41}$$

where  $u' > 0$  above this curve (larger values of  $p$ ). The curves  $q' = 0$  and  $p' = 0$  attract the flow in the  $(p, q)$  plane. The curve  $q' = 0$  is independent of  $u$  and  $p$  and it disappears for **A** solutions ( $\tau \rightarrow 0$ ). However if  $x = 2$  in the **A** solution, then  $q' = 0$  exactly. The curve  $p' = 0$  is independent of  $\tau$  and is a hyperbola in the  $(p, q)$  plane, asymptoting the lines  $p = \pm q$ . When  $u \rightarrow 0$ , the hyperbola approaches these lines. The curve  $u' = 0$  can be different conical curves depending on the sign of the coefficient of the  $q^2$  term and therefore on the value of  $\tau$ . For  $2 \tanh^2 \tau = 1$ , the curve is a parabola passing through  $(p = x/\sqrt{2}, q = 0)$  and  $(p = 0, q = 0)$ .

---

<sup>8</sup> The analysis for  $x > 2$  can be repeated using Seiberg duality  $N_c \rightarrow N_f - N_c$ ,  $H \leftrightarrow G$  that in this notation corresponds to  $p \rightarrow (x-1)p$ ,  $q \rightarrow -(x-1)q$ ,  $u \rightarrow (x-1)^3 u$  and then use  $\tilde{x} = x/(x-1)$ , so  $1 < \tilde{x} \leq 2$ .

For larger values of  $\tau$ , the curve is an ellipse, passing through  $(p = x/\cosh \tau, q = 0)$  and  $(p = 0, q = 0)$ . When  $\tau \rightarrow \infty$ , the ellipse collapses to the point  $(p = 0, q = 0)$ , so there is no region where  $u' < 0$ .

For smaller values of  $\tau$ , the curve is a hyperbola. For low values of  $\tau$ , one of the branches passes through  $(p = 0, q = 0)$  but lies below the  $p = \pm q > 0$  lines, so it does not affect the physical solutions. The other branch passes through the point  $(p = x/\cosh \tau, q = 0)$ . As we increase  $\tau$ , the two branches come closer until they merge at

$$\sinh^2 \tau = \frac{4(x-1)}{x^2}, \quad (4.42)$$

and form a new hyperbola where now the relevant branch passes through the points  $(p = 0, q = 0)$  and  $(p = x/\cosh \tau, q = 0)$ .

For solutions of type **A** ( $\tau = 0$ ), when  $N_f = 2N_c$  ( $x = 2$ ) the  $p' = 0$  curve is a fixed line for the flow in the plane defined by constant  $u$  (Fig. 1). The points where this curve intersects the curve  $u' = 0$  are fixed points of the three-dimensional system, and they correspond to the exact  $N_f = 2N_c$  solution found in [9, 12]. In terms of the original functions appearing in the metric  $h = (p+q)/4$ ,  $g = (p-q)/4$ , the set of fixed points form a curve in the  $(h, g, u)$  space that can be parametrized as

$$(h, g, u) = \left( h, \frac{h}{4h-1}, \frac{4h^2}{4h-1} \right), \quad \infty > h > 1/4. \quad (4.43)$$

The fixed line only exists for  $N_f = 2N_c$ . In the notation of [9, 12], the line of fixed points is parametrized by  $\xi = 1/h$  (see eq. (4.5)).

We can easily extend the analysis for a smaller number of flavors  $N_f \leq N_c$ . There are two main effects as  $x \rightarrow 0$ . The first is the lift of the attractor region (4.40) towards infinity, so it disappears completely when  $x = 0$ , and then  $p' > 0$  on the  $(p, q)$  plane. The second is the modification of the  $u' = 0$  curve. When  $x = 1$ , the merging of the two branches of the low  $\tau$  hyperbola occurs at  $\tau = 0$ . Then, the  $u' = 0$  curve is a straight line  $p = q + 1$  on the physical region and also at the border  $p = -q > 0$ . The region  $p > q + 1$  corresponds to  $u' > 0$ , while  $q + 1 > p > q > 0$  corresponds to  $u' < 0$ . When  $x < 1$ , the  $u' = 0$ ,  $\tau = 0$  curve is a hyperbola passing through the points  $(p = 0, q = 0)$  and  $(p = x, q = 0)$  and asymptoting  $p = q + 1$ .

In Section 4.3 we have presented the possible asymptotic behavior of solutions as a function of the radial coordinate  $\rho$ . We will comment on them under the perspective of flows in the dynamical system defined by (4.39).

Let us start with the UV behavior  $\rho \rightarrow \infty$ . As we have already commented, both **N** flow towards the  $\tau = 0$  fixed plane, where **A** solutions live, so they have common asymptotics. There are two possible classes of asymptotics (see Section 4.3.1).

- For class II all functions grow exponentially, so this corresponds to a situation where the flow is in the  $u' > 0$  region and the attractor  $p' = 0$  moves towards infinity. The

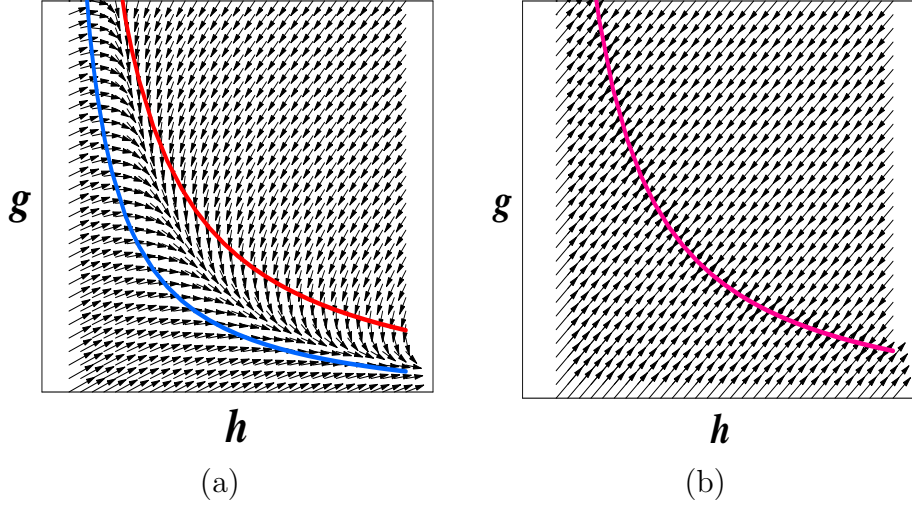


Figure 1: We represent the flow in the  $(p, q)$  plane at fixed  $u$  and  $\tau = 0$  (type **A** solutions). The physical region is above the black lines  $p = \pm q > 0$ . (a) The RG flow for  $N_c < N_f < 2N_c$ . The  $p' = 0$  line (red) acts as an attractor for the flow, the dark lines are the limits of the attractor region  $h' = 0$  and  $g' = 0$ . The  $u' = 0$  line (blue) has also been represented. (b) The RG flow for  $N_f = 2N_c$ . The  $p' = 0$  (red) and  $u' = 0$  (blue) lines have been represented. There are two fixed points of the flow at the intersection of the two curves.

attractor  $q' = 0$  will move towards positive or negative values of  $q$  depending on the value of  $N_f$ . The flow will try to reach the intersection of both. This is a very generic situation.

- Class I solutions, on the other hand, correspond to cases where the flow is in the  $u' < 0$  region, so the  $p' = 0$  attractor is dragged towards the lines  $p = \pm q$ . Generically, the flow would hit the lines and the solution will end at a finite value of the radial coordinate, so the corresponding solutions will have a boundary. Class I solutions are very special, because they correspond to a situation where  $u' \rightarrow 0^-$  asymptotically, so they are just marginal solutions between class II solutions and solutions with boundary. In the special case of  $N_f = 2N_c$  there is no flow in the  $q$  direction, so the flow will hit the fixed line  $p' = 0$ ,  $u' = 0$ . Notice that this is still a very special situation, usually the flow will miss the fixed line and go to a class II solution.

Let us examine now the IR behavior. Depending on the solution we can have  $\rho \rightarrow 0$  (for either  $\rho_0 < 0$  or  $\rho_0 = 0$ ) or  $\rho \rightarrow -\infty$ . The first case corresponds to type II and III, while the last corresponds to type I (see Section 4.3.2). In terms of the dynamical flow, the meaning is the following

- Type I: It corresponds to flows coming from infinity in the  $(p, q)$  plane, starting at the fixed subspace  $u = 0$ . Notice that they are always in the  $u' > 0$  region for **N** solutions or for **A** solutions when  $N_f > N_c$ . However, when  $N_f \leq N_c$ , the  $u' = 0$  curve on the  $(p, q)$

plane is different and can go to infinity. When  $N_f = 2N_c$  it is also possible to have flow starting at the fixed line  $p' = 0$ ,  $u' = 0$ , an example of this is the exact solution (4.7), that then runs into class II UV asymptotics. Notice that this is possible because the fixed line is not an attractor in all directions, it rather corresponds to a ‘saddle point’ in the flow.

- Type II and III: If  $\rho_0 < 0$ , they correspond to flows starting on the  $p = \pm q$  lines, with type III corresponding to the point  $p = q = 0$ . If  $\rho_0 = 0$ , then solutions have to start on the  $q' = 0$  line ( $q = 0$ ) at  $\tau = \infty$ .

A final comment is on the enlargement of the  $u' < 0$  region for  $N_f \leq N_c$ . Flows on that region are more likely to hit the  $p = q$  line, thus corresponding to a solution with boundary. From the field theory perspective this may correspond to the runaway behavior of the  $\mathcal{N} = 1$  SQCD theory when an ADS superpotential is generated. In the theories we consider here there is also a quartic superpotential, so the theory can show a different behavior.

## 5 Physics of the new solutions

In this Section we will study some aspects of the non-perturbative physics encoded by the new solutions we presented in Section 4. We will start with general comments, then we concentrate on aspects that involve the IR physics, hence using the IR (small  $\rho$ ) expansions derived in Section 4, and then we will move into some observables computed with the UV expansions ( $\rho \rightarrow \infty$ ). Some of the computations we will perform to calculate non-perturbative effects, have been thoroughly checked in the past, using different backgrounds with or without flavors; in those cases, we will be brief in our presentation. In contrast we will give details when new material is presented.

### 5.1 General Comments on the Field Theory

Let us first study an aspect that does not refer to the UV or the IR of the QFT, that is Seiberg duality [17].

It is by now well understood that for the particular field theories in the class that we are considering see eq. (2.4), Seiberg duality is not just the infrared equivalence of two theories, but the duality is valid all along the flow. This peculiar behavior is due to the presence of the quartic term in the quarks. Basically the argument is that when Seiberg dualized,

$$W \sim \kappa(Q\tilde{Q})^2 = \kappa MM \rightarrow W_{dual} \sim \kappa MM + \frac{1}{\mu} \tilde{q} M q, \quad (5.1)$$

where  $M$  is the meson superfield and  $q, \tilde{q}$  are the dual quarks. Now, the presence of the  $M^2$  term-a mass term- allows us to integrate out the meson, leaving us with a superpotential of



the form

$$W_{dual} \sim \frac{1}{\kappa} (q\tilde{q})^2, \quad (5.2)$$

then, the theory is Seiberg dual to itself. This, by the way, is at the root of the working of the Klebanov-Strassler duality cascade. See the lectures [18] for lucid explanations.

How is the previous discussion reflected by our backgrounds? This was discussed at length in [12]. Here, we can briefly mention that our generalized BPS eqs. (3.12) and our Hamiltonian eq. (3.8) do indeed show an interesting symmetry. Indeed,

$$P \rightarrow P, \quad Q \rightarrow -Q, \quad \tau \rightarrow \tau, \quad Y \rightarrow Y, \quad \sigma \rightarrow -\sigma, \quad N_c \rightarrow N_f - N_c, \quad (5.3)$$

is an invariance of the equations, the Hamiltonian and the Hamilton-Jacobi principal function in eq. (3.10). This implies that if we find one solution, we have by replacement in eq. (5.3) found another one with the correct relations between color and flavor groups.

The  $U(1)$  R-symmetry of the QFT is associated with translations in the angle  $\psi$ . The breaking of the symmetry will be briefly discussed later, but this point together with the matching of global anomalies does not present any subtlety aside from what was discussed in [12].

As was proposed in the paper [23] the gaugino bilinear is related to the function  $b(\rho)$  that appears in the RR three form. We will keep that identification that will allow us to write an energy-radius relation (in the UV)

$$e^{\frac{2\rho}{3}} = \frac{\mu}{\Lambda}, \quad (UV, \rho \rightarrow \infty). \quad (5.4)$$

Using the information above, one can easily obtain that the solutions whose UV asymptotics is given in eqs. (4.15)-(4.16)-(4.17) represent the field theory described in Section 2, once a dimension six operator is added. The addition of this irrelevant term in the Lagrangian (like in a related example of [19, 20]) dramatically changes the UV of the field theory leading to a solution that is ‘away’ from the near horizon of the D5 branes. We believe that this irrelevant operator is related to the gauge sector, for instance  $O_6 = (\mathcal{W}_\alpha \mathcal{W}^\alpha)^{3/2} \sim (F_{\mu\nu})^3$ , or  $O_6 \sim (D_\mu F_{\nu\rho})^2$ , though there could be also operators related to the KK multiplets like  $O_6 = |\Phi_k|^6$ , etc. Our proposal that the irrelevant term is made out of operators transforming in the adjoint of the gauge group is due to the fact that the asymptotic eqs. (4.15)-(4.16)-(4.17) exist also in the case of  $N_f = 0$ . The field theory aspects in the flavor-less case were studied and the solution was (numerically) found in Section 8 of [9].

Let us proceed by studying aspects of the IR of the field theory, encoded in the backgrounds of Section 4.

## 5.2 Physics in the Infrared

As stated above, we will start by studying non-perturbative effects that the backgrounds encode in the small radius region ( $\rho \rightarrow 0$ )<sup>9</sup>. We will also comment on Wilson, 't Hooft and dyon loops, domain walls and the behavior of the quartic coupling in the field theory described in eq. (2.4). Finally we will give a general analysis to find what are the conditions that determine if an object is screened in the dual theory.

### 5.2.1 Enhancement of the flavor group

In this section we describe something that applies equally to the solutions discussed here and the ones in [9] and [12]. We can ask the following question: what is the dual flavor group? The first answer may be that the group is  $SU(N_f)$ , but a better analysis shows that actually the flavor branes are separated-this is an effect of the smearing-so, a more likely answer is that the group is  $U(1)^{N_f}$ . Many dynamical aspects will not depend on this and it may happen that if the distance between branes is vanishing (for example in the IR) then, the strings stretching between flavor branes become massless and the  $SU(N_f)$  symmetry is recovered in that regime of energies. The fact that the F1-strings stretching between flavor branes have a mass has important consequences regarding the existence-or not- of diagrams that could likely correct the BI-WZ action that we used for the flavor branes.

To analyze this, we will proceed as follows. We will compute the volume of the space  $\Sigma_4$  on which we smear the flavor branes. We will divide this by  $N_f$  and associate a given “volume per flavor brane”. We will then propose that the distance between flavor branes is given by the fourth root of such ‘volume per flavor brane’ and estimate the mass of the F1-strings as the string tension times this quantity.

We also need to estimate the proper energy  $E_{proper}$  of an object in the bulk, given the energy of the dual object in the field theory  $E_{QFT}$ . The relation is

$$E_{QFT} = \sqrt{g_{tt}} E_{proper} = e^{\phi/2} E_{proper}. \quad (5.5)$$

Due to the dependence of the dilaton on the radial position, if we keep the QFT energy fixed, then the proper energy of the mode must vary. If the proper energy grows very fast in the IR, it could be possible to excite strings between different branes even if they are at a finite separation, leading to an enhancement of the flavor group to  $SU(N_f)$ .

Let us see the computation. We focus first on type **A** solutions. In the string frame, the induced metric on the four cycle  $\Sigma_4$  is,

$$ds_{\Sigma_4}^2 = e^{\phi} \left[ H(\rho)(d\theta^2 + \sin^2 \theta d\varphi^2) + G(\rho)(d\tilde{\theta}^2 + \sin^2 \tilde{\theta} d\tilde{\varphi}^2) + Y(\rho)(\cos \theta d\varphi + \cos \tilde{\theta} d\tilde{\varphi})^2 \right]. \quad (5.6)$$

---

<sup>9</sup>For convenience we are taking the IR region to be at  $\rho \rightarrow 0$ , but as explained above, in general it is at  $\rho \rightarrow \rho_{IR}$

So, the volume to be computed is

$$V_4 = \int d\theta d\tilde{\theta} d\varphi d\tilde{\varphi} \sqrt{g_{\Sigma_4}} = 4\pi^2 e^{2\phi} \sqrt{HG} \int d\theta d\tilde{\theta} \sqrt{HG \sin^2 \theta \sin^2 \tilde{\theta} + YG \cos^2 \theta \sin^2 \tilde{\theta} + HY \cos^2 \tilde{\theta} \sin^2 \theta}. \quad (5.7)$$

So, we need to look for the values of this integrals in the IR. Indeed, in the UV, is clear from the asymptotic expansions, that this integral will indeed diverge, making the mass of the F1s to diverge. The group is then broken to  $U(1)^{N_f}$  in the UV. In the IR, the analysis depends on the type of solution we consider. Let us first analyze this for the three types of IR type **A** solutions of the  $SU(N_c)$  dual background in [12]. They were called type I, II, and III IR behaviors. In type I and II, the previous volume is constant, indicating that there will be a fixed separation between flavor branes. In the type II case, the dilaton goes to a constant, so at very low energies, the strings between branes will decouple, breaking the flavor group to  $U(1)^{N_f}$ . In type I the dilaton vanishes, so the proper energy grows unbounded in the IR and the group gets enhanced to  $SU(N_f)$ . In type III, we observe that the volume vanishes, also enhancing the flavor group.

A similar analysis can be done for the type **N** solutions, using the determinant of the induced metric in the far IR. Given the expansion for type **N** solutions in Section 4, we obtain that for type I and II the volume diverges at  $\rho = 0$  and the dilaton is constant. This implies that non-diagonal strings are very massive and do not contribute to processes in the IR physics. The flavor group is then broken to  $U(1)^{N_f}$  by the solutions. This is basically the reason why in the Introduction we proposed different couplings between quark superfields and adjoint KK modes see eq. (2.3)-the difference may be just a phase factor for different positions of branes in  $\Sigma_4$ . In type III solutions on the other hand, the volume vanishes in the IR and the flavor group gets enhanced to  $SU(N_f)$ .

### 5.2.2 String-like objects

It is known that the holographic dual to the Wilson loop corresponds to a fundamental string describing the 4d Wilson loop in the Minkowski directions, exploring the radial directions and ending on a brane at spatial infinity [21], while in the type of backgrounds that are not S-duality invariant, the 't Hooft loop is computed by a D3 brane wrapping a topologically trivial two-cycle. These string-like objects should have only a finite length due to the screening from fundamental matter.

In the case of type **A** backgrounds, it was shown that depending on the IR of the solution, different vacua of the QFT present distinctive behavior for Wilson and 't Hooft loops, see Sections 4.4 and 4.5 of the paper [12] for a detailed analysis. Indeed (always in the case of type **A** backgrounds) it was seen that in type I solutions the Wilson loop tension vanishes, while it is finite for type II and III. On the other hand, the tension of the 't Hooft loop vanishes

only for type III solutions. Therefore, type I solutions correspond to a Higgs phase, while type III correspond to confined phases. In type II both quarks and monopoles are confined, so it should correspond to a oblique confined phase, where dyons with both electric and magnetic charge should be free.

**Wilson loops** In a QFT with fundamental matter, the QCD-string can break, basically due to pair creation. This effect is offset if one is working in the quenched approximation ( $\frac{N_f}{N_c} \sim 0$ ). Our backgrounds do indeed capture screening effects as was seen in [9, 12], by the existence of maximal length for the quark-antiquark separation. Nevertheless one can get an *intuitive* idea about the way a quark pair interacts by looking at the QCD string tension in the far IR. This is computed in the string duals by the value of the quantity (see [22] for a summary)

$$T_{QCD} = \sqrt{g_{tt}g_{xx}}|_{\rho_{IR}}. \quad (5.8)$$

We can see that this quantity is non-zero for the three types of IR asymptotics of type **N** backgrounds. Indeed, it is just proportional to the value of  $e^{\phi(\rho_{IR})}$ ; this indicates that at low energies, the pair feels a linear potential<sup>10</sup>. Contrast this with the type **A** backgrounds where for asymptotics of type I the force was vanishing. The main difference between type **A** and type **N** backgrounds resides in the existence of the functions  $a(\rho)$ ,  $b(\rho)$  that as was discussed in the literature [23], can be associated with the formation of a gaugino condensate. This seems to be at the root of the different behavior for the QCD string tension.

Let us now study another gauge invariant observable, the 't Hooft loop.

**'t Hooft loops** As discussed previously in the literature, for backgrounds that are not invariant under S-duality like the present ones, the 't Hooft loop is computed via a D3 brane that wraps a two cycle in the internal space. The manifold on which the D3 is proposed to extend is,

$$\Sigma_4 = [x, t, \theta = \tilde{\theta}, \varphi = 2\pi - \tilde{\varphi}, \psi = \pi, \rho(x)]. \quad (5.9)$$

The induced metric for a D3 on that manifold is

$$ds_{ind}^2 = e^{\phi} \left[ -dt^2 + dx^2(1 + e^{2k}\rho'^2) + (e^{2h} + \frac{e^{2g}}{4}(a-1)^2)(d\theta^2 + \sin^2\theta d\varphi^2) \right]. \quad (5.10)$$

After integrating over the internal manifold, the 'effective string' (representing the 't Hooft loop) has an action,

$$S_{eff} = 4\pi T_{D3} \int dx e^{\phi} (e^{2h} + \frac{e^{2g}}{4}(a-1)^2) \sqrt{(1 + e^{2k}\rho'^2)} = \int dx T_{eff} \sqrt{(1 + e^{2k}\rho'^2)}. \quad (5.11)$$

This effective tension  $T_{eff}$  plays a similar role to that of the QCD-string tension in the previous subsection. This object can be studied in the IR (again, keeping in mind that the 't Hooft

---

<sup>10</sup>Of course, at even longer distances, screening takes place and the potential changes.

loop will also present screening effects at even lower energies) to get an intuitive idea of the force, at low energies, between a pair of monopoles.

We obtain using the type **N** IR expansions that for type I and type III backgrounds the effective tension vanishes, leaving the monopoles free. In contrast, for type II backgrounds the effective tension is constant, leading to a linear potential between them.

It is a natural next step, to compute the tension between a pair of dyons. We are not aware of a concrete computation to calculate this effect, then we will propose one. So, we propose that the dyon-anti-dyon is represented by a D3 brane extended on  $(t, x)$  wrapping the topologically trivial cycle,

$$\theta = \tilde{\theta}, \quad \varphi = 2\pi - \tilde{\varphi}, \quad (5.12)$$

and that is charged by the presence of a world-volume electric field  $F_{tx}$ . We will show that for a particular value of the electric charge, the tension of the dyon loop indeed vanishes, in agreement with the oblique confinement picture. We will do this in the case of the simpler type **A** backgrounds for illustrative purposes.

**Computing dyon-antidyon loops** Since, as stated above, we believe this is new material, we will be more detailed. We study the proposal in type **A** and then (briefly) in type **N** backgrounds. In type **A** configurations, the induced metric, RR three form and world-volume fields on the D3 brane described around eq. (5.12) is (the brane also extends in the radial direction  $\rho(x)$ ),

$$\begin{aligned} ds_{ind}^2 &= e^\phi \left[ -dt^2 + (1 + 4Y\rho'^2)dx^2 + (H + G)(d\theta^2 + \sin^2\theta d\varphi^2) \right], \\ F_{tx} &= \partial_t A_x, \\ F_3 &= d \left[ \frac{(2N_c - N_f)}{4} (\psi - \psi_0) \sin\theta d\theta \wedge d\varphi \right]. \end{aligned} \quad (5.13)$$

We then compute the Born-Infeld part of the action,

$$\det[g_{ab} + 2\pi F_{ab}]^{1/2} = e^{2\phi} (H + G) \sin\theta \sqrt{1 + 4Y\rho'^2 - 4\pi^2 e^{-2\phi} F_{tx}^2}. \quad (5.14)$$

After integrating over the sphere directions and considering the WZ term that appears because of the presence of a nontrivial  $C_2$  on the worldvolume, the action for the D3 brane is,

$$S = -4\pi T_{D3} \left[ \int dx e^\phi (H + G) \sqrt{1 + 4Y\rho'^2 - 4\pi^2 e^{-2\phi} F_{tx}^2} - \Theta F_{tx} \right], \quad (5.15)$$

where  $\Theta$  is given by  $\Theta = \frac{2N_c - N_f}{4} (\psi - \psi_0)$ . We then compute the eq. of motion for the gauge field, obtaining

$$F_{tx} = - \frac{e^\phi (\Theta - \Theta_0) \sqrt{1 + 4Y\rho'^2}}{2\pi \sqrt{4\pi^2 (H + G)^2 + (\Theta - \Theta_0)^2}}, \quad (5.16)$$

where  $\Theta_0$  is an integration constant. Replacing back into the action (5.15), we get

$$S = -4\pi \int dx T_{eff} \sqrt{1 + 4Y\rho'^2}, \quad (5.17)$$

which is the action for an ‘effective string’ that is what we consider to be the QCD string (for dyons), with tension given by

$$T_{eff} = 2\pi T_{D3} e^\phi \frac{[(H + G)^2 + \Theta(\Theta - \Theta_0)]}{\sqrt{4\pi^2(H + G)^2 + (\Theta - \Theta_0)^2}}. \quad (5.18)$$

We can see that the dyon-anti-dyon tension depends on the type of solutions we deal with. Always thinking in the case of type **A** configurations and using the asymptotic expansions derived in [12], we see that for type I the tension is always finite, so the dyons are confined. For type III the tension vanishes only when  $\Theta_0 = \Theta$  (which means that  $F_{tx} = 0$ ) or  $\Theta = 0$ . In both cases, the dyon has no electric charge and becomes a monopole. For type II theories, in the IR  $H + G \rightarrow |C|$  (constant). If we choose  $\Theta_0$  conveniently, then the tension of the dyon vanishes. So, for each of the type II solutions and its Seiberg dual, a dyon with a precise value of the electric charge is free. Notice that a shift in  $\Theta$  changes the value of the electric charge of the free dyon, so the oblique confined phase is different, this is what we expect from field theory.

Having finished with the possible string-like objects (to which we will come back for a more general analysis in Section 5.2.5), we would like to study now higher dimensional defects.

### 5.2.3 Domain-wall tensions

A domain wall in these backgrounds is represented by a D5 brane that wraps (in a SUSY preserving way) a three-cycle and extends in three of the four Minkowski directions, preserving  $SO(1, 2)$ . The proposed manifold on which the five brane extends was given in [9] is

$$\Sigma_6 = [t, x_1, x_2, \theta, \varphi, \psi]. \quad (5.19)$$

The effective tension of this (2+1)-dimensional object-that is after integration over the internal three manifold- is

$$T_{eff,DW} = 16\pi^2 e^{2\phi+k} \left( e^{2h} + \frac{e^{2g} a^2}{4} \right). \quad (5.20)$$

The IR behavior of this quantity is such that the effective tension is constant for type I and type III solutions, while is divergent for type II asymptotics. This means (in the case of type II solutions) that the domain-wall like object has to sit at a finite value of the radial coordinate, where its tension is minimized. We move now to the study of our last IR observable, the value of the quartic coupling in the different asymptotics.

#### 5.2.4 Quartic coupling

In the paper [12] a possible definition for the quartic coupling  $\kappa$  between quark superfields in the QFT has been proposed. The definition is just based on dimensional analysis and has the right UV beta function, but does not transform correctly under Seiberg duality. We would like to propose here a different definition with the right properties

$$\log \kappa^2 = \frac{Vol[\tilde{S}^3] - Vol[S^3]}{Vol[S^1] \times Vol[S^2]} = -\frac{e^{2h} + \frac{e^{2g}}{4}(a^2 - 1)}{e^{2h} + \frac{e^{2g}}{4}(a - 1)^2} = -\frac{Q}{P}e^\tau \quad (5.21)$$

For type **A** backgrounds  $\log \kappa^2 = (G - H)/(G + H)$ . Seiberg duality interchanges the two three spheres in the metric, that in (5.21) is seen as  $\kappa \rightarrow \kappa' = 1/\kappa$ . In self-dual configurations  $H = G$ ,  $\kappa = \kappa' = 1$ . Let us stress again that this choice is based on some general properties but we do not have a proof of it being the actual quartic coupling of the dual field theory.

Analyzing the behavior of this quantity in the case of type **A** and **N** backgrounds, we see that  $\kappa$  always goes to a constant value in the IR or UV. In type **A** III, **N** I and **N** III infrareds and in class II ultraviolet, it flows to the self-dual value  $\kappa = 1$ .

#### 5.2.5 Screening as string breaking

In the papers [9] and [12] it was explicitly shown (using the dual backgrounds) that due to the presence of the fundamental matter, different ‘loops’ (Wilson, ’t Hooft, dyon) may present a maximal length, indicating screening. In this Section, we would like to present a general study whose outcome is a *sufficient* condition for which a finite maximal length will occur (or not). Our analysis will not depend on a particular asymptotics. Let us emphasize that, while the existence of a maximal length implies screening, the existence of an infinite length does not imply confining behavior. We are considering the “connected” Wilson loop and its energy must be compared with the possible “disconnected” solution indicating creation of pairs and screening-for another example where the same comment applies see [25]. The energetically favorable solution will be physically realized.

Let us then examine the issue of the maximal length of string-like objects described by a string/brane extending on  $t$ ,  $x$  and wrapping the internal space. The bulk radial coordinate is  $\rho$ , and the profile of this object is described by the function  $\rho(x)$ . If the metric is

$$ds^2 = e^{\phi(\rho)}(-dt^2 + d\mathbf{x}^2 + 4Y(\rho)d\rho^2) + g_{\alpha\beta}(\rho, y)dy^\alpha dy^\beta, \quad (5.22)$$

with  $g_{\alpha\beta}$  the metric of the internal space, then the Lagrangian coming from the Born-Infeld-Wess-Zumino action, after integration over the internal directions  $y^\alpha$ , will have the general form,

$$\mathcal{L} = T(\rho)\sqrt{1 + 4Y\rho'^2}, \quad (5.23)$$

where  $T(\rho)$  is the effective tension of the brane that depends in general on the dilaton and the components of the metric in the directions of the internal space that the brane is wrapping. The usual analysis goes as follows: we first define the Hamiltonian, that is a conserved quantity

$$\mathcal{H} = \frac{\partial \mathcal{L}}{\partial \rho'} \rho' - \mathcal{L} = -\frac{T}{\sqrt{1 + 4Y\rho'^2}} \equiv -T_0. \quad (5.24)$$

If for some value of  $\rho = \rho_0$ <sup>11</sup> the profile ends smoothly  $\rho' = 0$ , then  $T_0 = T(\rho_0)$ . From this expression, it is straightforward to find

$$\frac{dx}{d\rho} = \frac{2\sqrt{Y}T_0}{\sqrt{T^2 - T_0^2}}. \quad (5.25)$$

In order to have a sensible (real-valued solution), we need that  $T > T_0$  over all the brane (we are assuming that all quantities are positive). So,

- If the tension is a growing function of the radial position  $T'(\rho) \geq 0^+$  as  $\rho \rightarrow \rho_0$ , then the solution exist for  $\rho > \rho_0$ .
- If the tension is a decreasing function of the radial position  $T'(\rho) \leq 0^-$  as  $\rho \rightarrow \rho_0$ , then the solution exist for  $\rho < \rho_0$ .

Physically, this means that the object extends towards values where its tension is smaller. In the backgrounds we are considering we have usually dilaton factors that grow exponentially in the tension as  $\rho \rightarrow \infty$ , hence the branes will ‘hang’ from the UV towards the IR. However, close to the singularity, some functions of the metric grow and some functions decrease, so in principle there could be objects that cannot go beyond some finite value of the radial coordinate, and there could be objects that extend from the singularity towards the UV, although they cannot go arbitrarily far. When this happens, the objects will have a finite tension, the region where  $T' = 0$  acting as an effective IR wall for them. Let us analyze these cases in detail.

The simpler case is when there is no IR wall and the far IR is at  $\rho = 0$ , so we can use the asymptotic expansions. Let us assume that the tension of the object vanishes in the IR as some power  $T \sim \rho^b$ ,  $b > 0$ . Also, we assume that the function  $Y$  also has some power-like behavior, but can vanish, be a constant or blow up like  $Y \sim \rho^a$ . Then, the equation of motion in eq. (5.24) reads,

$$\frac{dx}{d\rho} \sim \frac{\rho^{a/2}\rho_0^b}{\sqrt{\rho^{2b} - \rho_0^{2b}}}. \quad (5.26)$$

---

<sup>11</sup>Notice that in this Section  $\rho_0$  denotes the turning point of the string-like object and should not be confused with the quantity defined earlier  $\rho_o$  that is the lowest value of the coordinate  $\rho$  in the background. The context will make this clear.



Close to the tip,  $\rho \simeq \rho_0(1 + \epsilon)$ . We can integrate trivially on  $\epsilon$  to find a square root behavior, so the profile at the tip looks like

$$x(\rho) \sim \rho_0^{(1+a)/2} \sqrt{\rho - \rho_0}. \quad (5.27)$$

The criterion we will adopt to quickly evaluate the ‘length’ of the putative loop is the following: the factor in front of the square root determines how “open” is the configuration. If the configuration is to reach infinite length in the  $x$  direction when the object explores the far IR of the dual theory, then this factor should diverge when  $\rho_0 \rightarrow 0$ . In the case at hand, this implies that  $a < -1$ . This is indeed the case of the AdS string, where  $T \sim \rho^2$  and  $Y \sim \rho^{-4}$ .

The next possibility is to consider an object that has finite tension, so that  $T \sim T_* + c\rho^b$ ,  $b > 0$ ,  $c > 0$  (this is necessary to have  $T' > 0$ ) when  $\rho \rightarrow 0$ . Now the equation of motion is a bit different

$$\frac{dx}{d\rho} \sim \frac{T_* \rho^{a/2}}{\sqrt{(T_* + c\rho^b)^2 - (T_* + c\rho_0^b)^2}} \sim \frac{T_*^{1/2} \rho^{a/2}}{\sqrt{\rho^b - \rho_0^b}}. \quad (5.28)$$

The profile at the tip now looks like

$$x(\rho) \sim T_*^{1/2} \rho_0^{(1+a-b)/2} \sqrt{\rho - \rho_0}. \quad (5.29)$$

For example, we can analyze the previous expression for the case of the flavorless solution ( $N_f = 0$ ) in eq. (4.2), and check that this gives  $a = 0, b = 2$  implying an infinite length. As another example, we analyze the IR of type **A** backgrounds. In order to have an object that can extend to infinite length, we should have that  $b > 1/2$  in type II and  $b > 1/3$  in type III. Notice that the monopoles and dyons have  $b = 1/2$  and  $b = 1/3$ , so they are screened objects.

Suppose now, that we have an object that feels an effective IR wall at  $\rho = \rho_*$  where the tension is minimized, so  $T'(\rho_*) = 0$  and  $T''(\rho_*) > 0$ . We are assuming that the background at this point is smooth, and the functions of the metric do not vanish, so  $Y$  and the tension admits Taylor expansions

$$\begin{aligned} T &= T_* + T_2(\rho - \rho_*)^2 + \dots, \quad T_2 > 0, \\ Y &= Y_* + Y_1(\rho - \rho_*) + \dots \end{aligned} \quad (5.30)$$

We take a configuration where the object ends smoothly at some point  $\rho_0 > \rho_*$  and consider the limit  $\rho_0 \rightarrow \rho_*$ . It is convenient to use the coordinate  $r = \rho - \rho_*$  and  $r_0 = \rho_0 - \rho_*$  so eq. (5.25) becomes,

$$\frac{dx}{dr} \sim \frac{\sqrt{Y_* T_*}}{\sqrt{(T_* + T_2 r^2)^2 - (T_* + T_2 r_0^2)^2}} \sim \sqrt{\frac{Y_* T_*}{T_2}} \frac{1}{\sqrt{r^2 - r_0^2}}. \quad (5.31)$$

Close to the tip,  $r \simeq r_0(1 + \epsilon)$ . We can integrate trivially on  $\epsilon$  to find a square root behavior, so the profile at the tip looks like

$$x \sim \sqrt{\frac{Y_* T_*}{T_2}} r_0^{-1/2} \sqrt{r - r_0}. \quad (5.32)$$

Therefore, when the tip of the object comes close to the IR wall  $r_0 \rightarrow 0$ , the length diverges.

It should be interesting to extend the analysis of this Section to the case in which we have massive quarks; because it could provide more hints on the first order phase transition in terms of the mass- that occurs for the connected part of the Wilson loop, see [25]. Also, it may be interesting to use the results of this Section to improve models of AdS/QCD to incorporate screening effects. To close, let us mention that it should be nice to provide a link between the formalism developed here and the ideas presented in [24].

Let us now move to study aspects of the UV of the field theory, using the large  $\rho$  region of the solutions.

### 5.3 Physics in the Ultraviolet

Let us briefly study how some aspects of the UV field theory are captured by the backgrounds. R-symmetry anomalies will be treated in a sketchy way. The results are similar to those in [9] and [12]. The study of beta functions will be more detailed because it will shed light on a long-standing puzzle.

The way the R-symmetry anomaly will work, as explained at length in [9] and [12] does not change here. Basically this is because it only depends on the form of the RR field  $F_3$  when restricted to a particular manifold-see eq. (5.12)-considered at large values of the radial coordinate. The result is not changed when switching between the different UV solutions we found in Section 4.

Let us then concentrate on the beta function of the field theory as computed by the two different UV expansions. We will present details, as this will teach us something interesting about the field theory and its UV completion.

#### 5.3.1 Beta function

A definition of the gauge coupling in the dual QFT in terms of type IIB objects was given in [26]. Using that definition, we have an expression for the gauge coupling that reads,

$$\frac{8\pi^2}{g^2} = 2(e^{2h} + \frac{e^{2g}}{4}(a-1)^2) = e^{-\tau} P. \quad (5.33)$$

As mentioned above, the relation between the scale of the QFT and the radial coordinate was also proposed in [23] and [26]. This relation comes by associating the functions  $a(\rho)$  and/or  $b(\rho)$  with the VEV of the gaugino condensate. One gets,

$$e^{\frac{2g}{3}} = \frac{\mu}{\Lambda}, \quad (UV, \rho \rightarrow \infty). \quad (5.34)$$

We have found two types of UV behavior. One of them was already studied in [9] and [12]. The other, for the solutions that asymptote to the conifold, was given in eqs. (4.15)-(4.16)-(4.17). For both asymptotics, the relation in eq. (5.34) above holds.

This allows us to compute the behavior of the beta function in the UV of the QFT. The results are that for the UV expansions presented in [9] and [12] we get (for  $N_f < 2N_c$  as explained in those papers) that

$$\beta_{\frac{8\pi^2}{g^2}} = \frac{3}{2}[2N_c - N_f], \quad (5.35)$$

that is the result expected from the NSVZ beta function, in the case in which the anomalous dimension of the quark superfield is  $\gamma_Q = -\frac{1}{2}$ . See [12] for various consistency checks on this value of the anomalous dimension.

Now, this is the puzzle: how is it possible that a theory with so many extra fields (in the UV, the extra modes should be integrated-in) presents a beta function that adjusts so well to the exact treatment of NSVZ. The comparison with the result obtained using the other possible UV solution will illuminate this point. This also applies to the case  $N_f = 0$  and we believe it answers the criticism to the papers [26].

**The beta function for a 6d theory** The set-ups described in the Introduction are the holographic duals of the six-dimensional field theories living on the D5 branes that wrap a 2-cycle of the deformed conifold. The dimensionful gauge coupling can be read from the D5 brane action  $g_6^2 = \alpha' g_s$ , and one can define the dimensionless coupling

$$\tilde{g}_6^2 = \mu^2 g_6^2 Z^2(\mu), \quad (5.36)$$

where  $\mu$  is some mass scale and  $Z(\mu)$  is the renormalization factor of the 6d coupling, analogous to a mass renormalization factor. The theory is perturbatively non-renormalizable, in the sense that radiative corrections can generate an infinite set of irrelevant operators as the theory flows to the UV. From a four-dimensional perspective, we can define a (dimensionless) coupling

$$g_4^2 = \frac{1}{L^2} g_6^2 Z^2(\mu), \quad (5.37)$$

where  $L^2 \sim N_c \alpha'$  is the size of the 2-cycle. In this case, the beta function is

$$\beta_{\frac{8\pi^2}{g_4^2}} = -\frac{16\pi^2}{g_4^2} \frac{d \log Z}{d \log \mu} \equiv -\frac{16\pi^2}{g_4^2} \gamma_{\tilde{g}}, \quad (5.38)$$

At one-loop in perturbation theory, the infinite tower of extra states (KK modes) would generate a power-like running of the coupling in the UV

$$g_4^2 \sim \frac{1}{(L\mu)^2}. \quad (5.39)$$

If we integrate out the KK modes, then an effective coupling for an irrelevant operator of dimension six would be generated, also modifying the running of  $g_4$ . As the energy scale

becomes of the order of the Kaluza-Klein mass, more irrelevant operators would enter, making the theory non-renormalizable.

However, a six dimensional theory could be renormalizable at a non-perturbative level. This is possible if there is a UV fixed point, as was shown in [27]. Looking at the beta function (see Kazakov's paper in ref. [27])

$$\beta_{\frac{8\pi^2}{\tilde{g}_6^2}} = -\frac{16\pi^2}{\tilde{g}_6^2}(1 + \gamma_{\tilde{g}}), \quad (5.40)$$

we see that a UV fixed point corresponds to an anomalous dimension  $\gamma_{\tilde{g}} = -1$ . On the other hand, the four dimensional coupling would have the same running as the one-loop perturbative result, suggesting that only a dimension six operator is generated, instead of an infinite set of irrelevant operators. We can take this as an indication that the UV theory is truly six dimensional, where such operator is marginal. It has been suggested that the marginal operator respects scale invariance but not the full conformal invariance of the theory.

We can now see how this picture emerges in the holographic description. Using the prescription of [26], we can see that the renormalization factor  $Z^2$  is proportional to the inverse size of the cycle. In the UV, the relation between the radial coordinate  $\rho$  and the scale  $\mu$  is given as explained above by eq. (5.34). Then, for the solutions asymptoting to the conifold, eqs. (4.15)-(4.16)-(4.17), we have

$$\log Z = -\frac{2}{3}\rho = -\log \frac{\mu}{\Lambda} \Rightarrow \gamma_{\tilde{g}} = -1. \quad (5.41)$$

So the growing solutions correspond to six dimensional theories flowing towards a UV fixed point. Let us examine the metric in the far UV, keeping only the leading asymptotics and using the coordinate  $2r = 3e^{2\rho/3}$ , the background in eq. (2.13) with the asymptotics (4.15)-(4.16)-(4.17) will read,

$$ds^2 = e^{\phi_0} \left[ dx_{1,3}^2 + \frac{4c_+}{3}(dr^2 + r^2 T^{1,1}) \right] \quad (5.42)$$

There is a scaling symmetry; we can rescale the coordinates  $r \rightarrow \lambda r$ ,  $x^\mu \rightarrow \lambda x^\mu$ . This changes the metric only by a conformal factor, that can be absorbed in the dilaton. Notice that the RG flow is independent of the value of the dilaton. There is also a self-similarity relation between different RG flows in the UV, since the transformation  $r \rightarrow \lambda r$ ,  $c_+ \rightarrow \lambda^{-2}c_+$ , leaves the metric invariant.

The special solutions with UV given in [12] -see eqs. (4.9)-(4.13)- have a rather different behavior. The renormalization factor goes as

$$\log Z \sim -\frac{1}{2}\log \rho \sim -\frac{1}{2}\log \log \frac{\mu}{\Lambda}. \quad (5.43)$$

This gives the anomalous dimensions and gauge couplings

$$\gamma_{\tilde{g}} = -\frac{1}{2\log \frac{\mu}{\Lambda}} \Rightarrow g_4^2 \sim \frac{1}{\log \frac{\mu}{\Lambda}}, \quad \tilde{g}_6^2 \sim \frac{\mu^2}{\log \frac{\mu}{\Lambda}}. \quad (5.44)$$

The running of the  $g_4$  coupling is exactly the one of a four dimensional  $\mathcal{N} = 1$  theory. This is quite surprising, since a six dimensional theory is showing a completely four dimensional behavior. From the string theory point of view, the six-dimensional solutions (5.42) look as a decompactification limit and actually the metric coincides with the metric ‘outside’ the near-horizon region of the 5-branes<sup>12</sup>. Notice that the dilaton goes to a constant, so the energy of probes in the gravitational background is simply proportional to the energy in the field theory-see eq. (5.5)- and the extra dimensions can be easily explored.

On the other hand, in the ‘four dimensional solutions’ eqs. (4.9)-(4.13) the dilaton factor in the proper energy formula (5.5) grows exponentially with the radial coordinate, while the relative size of the compact dimensions grows only linearly. Then, for an object with fixed energy in the field theory, the proper energy in the string dual is exponentially suppressed at larger values of the radial coordinate in such a way that the wavelength will grow much faster than the size of the compact dimensions. Therefore, it is very difficult to explore the extra dimensions and the theory remains four-dimensional even in the ultraviolet.

In the dual field theory, the interpretation is that these backgrounds correspond to RG flows where the six dimensional irrelevant operator is not generated, so they are renormalizable not only in the six dimensional sense but also in a four dimensional sense. All these comments are consistent and provide a different view on the findings in [28], where it was shown that the beta function of the unflavored background was unaffected by the KK modes. For a field theory argument that complements this one, see Appendix A.

Let us now analyze another contribution of this paper. The extension of these set-ups to the case of orthogonal groups.

## 6 The case of $SO(N_c)$ gauge group

In this Section we address some aspects of the version of SQCD described in Section 2.1, but in the case in which the gauge group is  $SO(N_c)$ . The best places where the field theory is described are, for the pure SQCD case in [31] and the paper [32] for the theory that is more like the one we are interested in. For a recent analysis of a field theory related to ours in non-critical strings set-up see [30].

The  $N = 1$  theory with orthogonal group has a rich phase structure, with fixed infrared conformal points for  $\frac{3}{2}(N_c - 2) < N_f < 3(N_c - 2)$  and a free magnetic dual description for  $N_c - 2 < N_f \leq \frac{3}{2}(N_c - 2)$ . For  $N_f < N_c - 2$  there is a dynamically generated superpotential of the ADS type, associated to gaugino condensation. For  $N_f = N_c - 4$  and  $N_f = N_c - 3$  there is also an extra branch with no superpotential. In the field theories described by the holographic duals we study, this picture is modified by the presence of a classical superpotential coming from the reduction from the six dimensional theory.

---

<sup>12</sup>As predicted to happen in a different context in the papers [19, 20]

The dual geometry is based on the same 5-brane construction wrapping a 2-cycle of the resolved conifold. In order to have an orthogonal gauge group, we introduce an orientifold O5 plane parallel to the 5-branes. This setup was studied in [29]; the case of symplectic groups was also addressed there. The orientifold has the effect of making a reflection on the orthogonal directions, so the  $S^3$  surrounding the 5-branes becomes a projective space  $S^3 \rightarrow RP^3$ . When we go from the brane picture to the geometric picture, this is shown in the periodicity of the coordinate  $\psi \sim \psi + 2\pi$ . The gauge group and the three form flux in the geometry depend on the type of orientifold. We can classify them according to their five-brane charge and discrete torsions for the NS three-form  $H_3$  and the RR one-form  $F_1$  [33, 34]. If the number of D-branes is  $2n$ , then we have,

orientifold	charge	$(H_3, F_1)$	group	$F_3$
$O5^-$	-1	$(0, 0)$	$SO(2n)$	$2n - 2$
$O5^+$	+1	$(\frac{1}{2}, 0)$	$Sp(2n)$	$2n + 2$
$\widetilde{O5}^-$	$-\frac{1}{2}$	$(0, \frac{1}{2})$	$SO(2n + 1)$	$2n - 1$
$\widetilde{O5}^+$	+1	$(\frac{1}{2}, \frac{1}{2})$	$Sp(2n)$	$2n + 2$

The geometry of the orientifolded construction is essentially the same, with even the same Killing spinors. The only differences are the change of topology of a three-cycle of the internal space and the amount of  $F_3$  form flux.

In the following and to avoid a cluttered notation, we will concentrate on type **A** backgrounds. It will be clear that the important features do extend also to type **N** configurations, in virtue of our general treatment of Section 3. The BPS equations for the  $SO(N_c)$  case can be read directly from the  $SU(N_c)$  case from eqs. (2.9)-(2.12) by replacing

$$N_c \rightarrow N_c - 2. \quad (6.1)$$

So they read

$$\begin{aligned}
H' &= \frac{1}{2}(N_c - 2 - N_f) + 2Y, \\
G' &= -\frac{(N_c - 2)}{2} + 2Y, \\
Y' &= -\frac{1}{2}(N_f - N_c + 2)\frac{Y}{H} - \frac{(N_c - 2)}{2}\frac{Y}{G} - 2Y^2\left(\frac{1}{H} + \frac{1}{G}\right) + 4Y, \\
\phi' &= -\frac{(N_c - 2 - N_f)}{4H} + \frac{(N_c - 2)}{4G}.
\end{aligned} \quad (6.2)$$

At this stage, we observe a couple of immediate things about Seiberg duality and R-symmetry anomalies. The BPS system (6.2) is *invariant* if we change in all the eqs.,

$$N_c \rightarrow N_f - N_c + 4, \quad H \rightarrow G, \quad G \rightarrow H, \quad (6.3)$$

leaving all the other functions and  $N_f$  untouched. In our eq. (5.3) and in the paper [12] a similar change to eq. (6.3) was identified as Seiberg duality (for the group  $SU(N_c)$ ). In the case of orthogonal gauge groups, the change is as in eq. (6.3) and coincides with what is known from [31].

Using the RR three form and restricting it to a topologically trivial two cycle (on which  $F_3 = dC_2$ ), we can compute the anomaly of the R-symmetry, that is initially identified with changes in the angle  $\psi$ . Indeed, one can see that as previously explained in [12] the partition function for a Euclidean D1 that wraps the topologically trivial cycle (5.12) is given by  $Z = Z_{BI} e^{\frac{i}{2\pi} \int C_2}$ , where  $Z_{BI}$  is the Born-Infeld part of the partition function. This implies that the partition function is invariant if,  $\psi \rightarrow \psi + \frac{2k\pi}{2N_c - 4 - N_f}$ , hence showing that the R-symmetry is broken to  $Z_{2N_c - N_f - 4}$ . The extension to type **N** backgrounds is immediate. In this case, spontaneous breaking of R-symmetry imply that there are  $2N_c - N_f - 4$  vacua.

The new and interesting thing is that, although the geometry is pretty much the same, the change in topology in the duals to theories with orthogonal groups allows new kind of objects corresponding to branes wrapping cycles of the internal space. We follow the discussion in [33], adapting it to our case. The internal space has topology  $RP^3 \times S^2$ , and due to the O5 orientifolding in the brane construction, D1 and D5 branes can be wrapped in  $RP^3$  according to the untwisted homology, while F1 strings, NS5 and D3 branes should be wrapped according to the twisted one. The non-trivial homology groups are  $H_0(RP^3, Z) = H_3(RP^3, Z) = Z$ ,  $H_1(RP^3, Z) = Z_2$ , while the twisted ones are  $H_0(RP^3, \tilde{Z}) = H_2(RP^3, \tilde{Z}) = Z_2$ . There could be further constraints on the kind of wrapping that is allowed due to the discrete torsions of the different types of orientifold. Let us list the new possibilities of wrapping cycles inside the three-fold.

- a) A string on  $RP^2 \subset RP^3$ : this gives a point-like object localized in space and time. Two objects of this kind can annihilate each other, since they are classified by the group  $H_2(RP^3, \tilde{Z}) = Z_2$ .
- b) A D1 brane on  $RP^1 \subset RP^3$ : this gives a particle in the non-compact space. This particle should be able to annihilate with another of the same kind, since it is classified by a  $Z_2$  group.
- c) A D3 brane on  $RP^2 \subset RP^3$ : a string-like object in the non-compact space, also classified by  $Z_2$ .
- d) A D3 brane on  $RP^2 \times S^2$ : a point-like object, classified by  $Z_2$ . By  $S^2$  we mean any 2-cycle, it could be contractible.
- e) A D5 brane on  $RP^1 \subset RP^3$ : an object with four spatial dimensions, classified by  $Z_2$ .
- f) A D5 brane on  $RP^1 \times S^2$ : a domain wall-like object, classified by  $Z_2$ .

g) An NS5 brane on  $RP^2 \subset RP^3$ : an object with three spatial dimensions, classified by  $Z_2$ .

h) An NS5 brane on  $RP^2 \times S^2$ : a string-like object, classified by  $Z_2$ .

The most interesting objects are b), c)/h) and f), that have some of the properties to be the duals of the gauge theory Pfaffian, spinor Wilson loop and domain walls.

Discrete fluxes can impose a constraint on the allowed branes from the following considerations: imagine we have a brane wrapping a cycle in the internal space that contains a subcycle with discrete flux on it. In our case the possibilities are to have a  $RP^2$  or  $RP^1$  subcycle. We could consider a brane or string wrapping this subcycle, that due to the discrete flux, will be seen as an instantonic contribution to the wavefunction of the larger brane, changing its sign. For instance, wrapping a string on a  $RP^2$  subcycle will give a factor

$$e^{i \int_{RP^2} B_2} = -1.$$

This would rule out the wrapping brane as a good state of the theory unless it can be compensated by a twisted bundle of the gauge field on the brane,

$$e^{i \int_{RP^2} \tilde{F}_2} = -1.$$

However, the possible twisted bundles of the gauge field are determined by the twisted cohomology of the cycle the brane is wrapping, and not by the twisted cohomology of the complete internal space. They do not need to be the same, so it could happen that it is not possible to construct a twisted gauge bundle on the subcycle. In this case, the change of sign of the wavefunction cannot be compensated and the wrapping brane is ruled out.

In the case at hand, the discrete fluxes do not impose constraints on the branes, since the twisted cohomologies of  $RP^2$  and  $RP^1$  are non-zero. This makes it more difficult to identify the right objects, compared to the  $AdS_5 \times RP^5$  case, where the topological restrictions coming from the discrete torsions distinguish objects that exist only in orthogonal  $SO(2n)$  or  $SO(2n+1)$  theories.

## 6.1 Spinor Wilson loop

In orthogonal theories it is possible to introduce external sources in the spinor representation. These sources are not screened by the matter content of the theory, so the dual object should be able to extend to infinite length. Spinor charges do not exist in unitary theories, so the spinor Wilson loop should correspond to a D3 on  $RP^2$ . We will see that this “no screening criterion” allows to confirm that this D3 brane is the right object.

We then study our possible candidates for a spinor Wilson loop represented as a D3 brane wrapping  $RP^2$ . A possible  $RP^2$  could be the cycle

$$\begin{aligned} \theta = \tilde{\theta}, \quad \varphi = \tilde{\varphi} = 2\pi - \psi/2 & \text{ if } 0 \leq \theta < \pi/2, \\ \theta = \tilde{\theta}, \quad \varphi = \tilde{\varphi} = \psi/2 & \text{ if } \pi/2 \leq \theta < \pi. \end{aligned} \tag{6.4}$$



We are taking here  $\psi$  to be defined over  $[0, 4\pi)$  but with the identification  $\psi \sim 4\pi - \psi$ . Notice that the cycle is non-orientable.

For the clarity of this exercise, we will restrict ourselves to the case in which our branes probe a type **A** background. In this case, the induced metric will be

$$ds_{D3}^2 = e^\phi \left( -dt^2 + (1 + 4Y\rho'^2)dx^2 + (H + G)d\Omega_2^2 + 4Y(1 \mp \cos\theta)^2 d\varphi^2 \right). \quad (6.5)$$

The effective tension is then

$$T_{D3}(\rho) = 4\pi T_3 e^\phi \sqrt{H + G} \int_0^{\pi/2} d\theta \sqrt{(H + G) \sin^2 \theta + 4Y(1 - \cos \theta)^2}. \quad (6.6)$$

The integration can be done analytically, the final result is

$$T_{D3}(\rho) = 4\pi T_3 e^\phi (H + G) \left( 2 - \sqrt{1 + y} + \frac{2y}{\sqrt{1 - y}} \log \left[ \sqrt{2} \frac{1 + \sqrt{1 - y}}{\sqrt{1 + y} + \sqrt{1 - y}} \right] \right). \quad (6.7)$$

Where we have defined  $y \equiv 4Y/(H + G)$ . We compute numerically the value of the tension for this particular D3, using the expansions given in [12]. It turns out that the tension of the D3 brane always has a minimum at a finite value of  $\rho$ . As we have explained in Section 5.2.5, this means that the D3 brane is not screened in any background, so it is a good candidate for a spinor Wilson loop.

The fact that this object exists in the backgrounds dual to theories with orthogonal gauge group and does not exist for unitary groups is a very stringent dynamical check of our proposal and of the whole set-up.

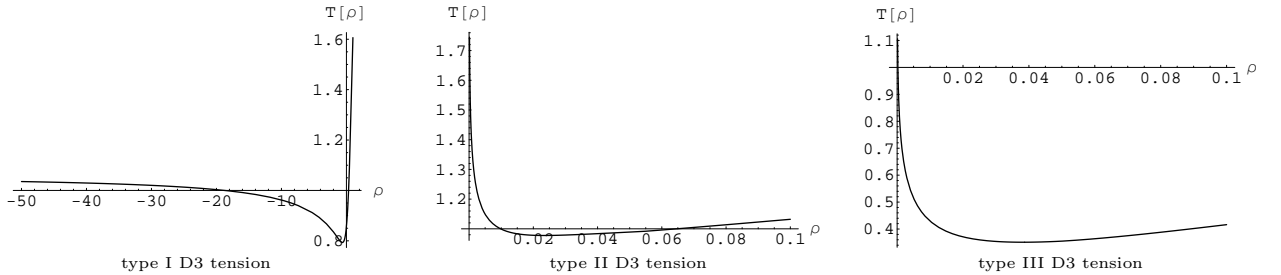


Figure 2: The tension of the D3 brane wrapping a  $RP^2$  cycle has a minimum as a function of  $\rho$  for the three types of **A** backgrounds, so the holographic dual object is not screened in any of them.

## 7 General comments, criticism and conclusions

In this Section we would like to present the outcome of many discussions we have had with different colleagues who asked challenging questions regarding various aspects of this line of research.

The first comment concerns the use of flavor branes represented by the Born-Infeld-Wess-Zumino Action in contrast with the color branes, encoded in the type IIB supergravity fields, in other words, we seem to treat differently the  $N_c$  “color branes” and the  $N_f$  flavor branes. The reason for this asymmetric treatment is that the two types of symmetries (flavor and gauge) are actually different. As is well known, a flavor symmetry is a symmetry of the physical spectrum (a ‘real’ symmetry), that can be broken; while the previous two properties do not apply to a gauge symmetry, that is just a way of indicating a redundant description. So, the main point is that an arbitrary flavor symmetry group  $SU(N_f)$  needs extra degrees of freedom to be added to the type IIB Action, this is what the Born-Infeld-Wess-Zumino Action is doing. This is the proposal of [10].

As a consequence, a fluctuation of the fields in the action of eq. (2.6) would imply a fluctuation of the gauge field on the brane, that is dual to a meson with flavor quantum numbers, showing the existence of meson-glueball interactions.

The reader may wonder what are the limitations of the line of research represented by [10] and papers that followed. Since the  $N_f$  ‘flavor branes’ are not deforming the spacetime (they are very few with respect to the color branes  $\frac{N_f}{N_c} \sim 0$ ) it is clear that some physical effect will not be reproduced by the String configuration. This can be seen diagrammatically in the dual field theory. Indeed, consider for example the scattering of two mesons in a theory with  $N_f$  flavors and  $N_c$  colors. A formula for the kinematical factor was produced in [35] for the scattering of  $n$  mesons, considering diagrams with  $w$  internal fermion loops (windows),  $h$  non-planar handles and  $b$  boundaries,

$$\langle B_1 \dots B_n \rangle \sim \left( \frac{N_f}{N_c} \right)^w N_c^{(2 - \frac{n}{2} - 2h - b)}. \quad (7.1)$$

Consider the case of scattering of two mesons  $n = 2$ . We see that diagrams like the first one in Figure 3 ( $w = h = 0, b = 1, n = 2$ ) scales like a constant  $N_c^0 \sim 1$ , the second diagram (with  $w = 1, h = 0, b = 1, n = 2$ ) scales like  $\frac{N_f}{N_c}$ , while the third one (with  $w = 0, h = 0, b = 2, n = 2$ , that is non-planar) goes like  $N_c^{-1}$ .

There are two possible scalings that can be taken to make contact with String theory:

- The one introduced by ’t Hooft [36] where  $\lambda = g^2 N_c$  is kept fixed while

$$g \rightarrow 0, \quad N_c \rightarrow \infty, \quad N_f = \text{fixed}. \quad (7.2)$$

In ’t Hooft’s scaling (7.2) the first diagram of the figure is dominant, the second one is suppressed like  $N_c^{-1}$  due to the quark loop and the third one, being non-planar is also suppressed like  $N_c^{-1}$ . We observe a suppression of both planar (with quark loops) and non-planar diagrams. It may be desirable to first sum up all planar diagrams and then consider non-planar ones as correction, this is achieved by the so called Veneziano expansion.

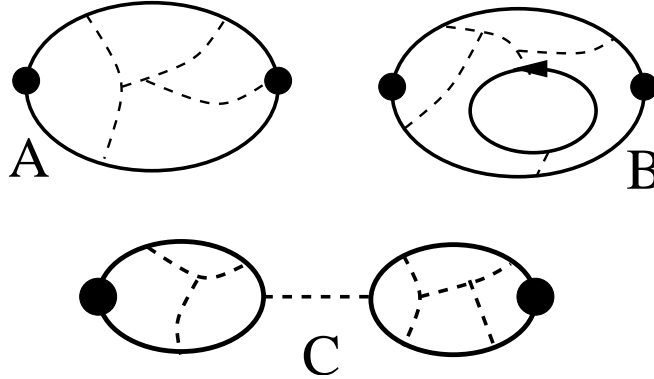


Figure 3: Diagram for a meson propagator, with two insertions of the meson operator ( $n = 2$ ) shown as thick points on the boundaries. The dashed lines are gluons that fill the diagram in the large  $N_c$  limit and the thick lines are quarks. A) Planar diagram with no internal quark loops ( $h = 0$ ,  $w = 0$ ), the scaling is  $\sim 1$ . B) Planar diagram with an internal quark loop  $h = 0$ ,  $w = 1$ , the scaling is  $\sim N_f/N_c$ . C) Non-planar diagram with no internal quark loops  $h = 0$ ,  $w = 0$ ,  $b = 2$ , the scaling is  $\sim 1/N_c$ .

- Veneziano [37] introduced another expansion ( called the topological expansion ) where the scaling is

$$g \rightarrow 0, N_c \rightarrow \infty, N_f \rightarrow \infty, \frac{N_f}{N_c} = \text{fixed}. \quad (7.3)$$

Obviously here we also impose that  $\lambda = g^2 N_c$  is fixed. In this scaling, the first two diagrams in the figure contribute at the same order and the one violating the OZI rule is suppressed.

The connection between these expansions and the way of adding flavors described above goes as follows. When we add probe branes to the background (hence not considering its backreaction), we are working in the 't Hooft scaling (7.2). The discussion above shows that some processes, those in which diagrams scaling like  $\frac{N_f}{N_c}$  are of importance (like screening, for example), will not be well captured by the 'probe approximation'.

We actually would like to impose the Veneziano scaling (7.3), that considers diagrams of the first two types, and hence is more likely to capture more physics. The way of doing this is by 'backreacting with the flavor branes'. That is to say, consider solutions to the eqs. of motion derived from the action (2.6).

Let us now comment on the effects of the smearing of flavor branes. As we hinted along the paper and more concretely in Section 5.2.1, the smearing seems to break the  $SU(N_f)$  flavor symmetry of the localized brane system to  $U(1)^{N_f}$ . This will obviously have influence on the dynamics, but also many aspects will remain unaffected (like global anomalies and their matching in dual descriptions, beta functions, Seiberg duality, etc). We would like to comment a bit more on the following related fact: if  $N_f$  is bigger than  $N_c$  and we take the

numbers such that we do not lose asymptotic freedom, it may be expected that there will be diagrams correcting the Born-Infeld action, that will be weighted by  $g_s N_f \sim \frac{N_f}{N_c} > 1$ . So, in principle, one should not trust the Born-Infeld-Wess-Zumino action, unless there is a way of understanding that such corrections will not change the tree-level form of the action. Here is where the smearing comes to help. The strings between flavor branes will typically be massive as we showed in Section 5.2.1. Even more, in the far UV, the separation between flavor branes is large (because the ‘internal four manifold  $\Sigma_4$ ’ is very large). In the IR (typically) something similar happens, with the ‘internal manifold’ having a fixed size and with low energies to excite the non-diagonal strings, see Section 5.2.1 for details. We believe that this makes the above mentioned corrections to be very suppressed or non-existent (a very similar argument was presented in [38]). As an aside, the fact that the system is SUSY suggests that even in the case of localized (in contrast to smeared) flavor branes the diagrams mediated by the non-diagonal strings will cancel each other (or the system would lose stability and be non-SUSY). The complementary possibility that would exclude what we are arguing here is that there is no *weakly coupled* String dual to a field theory with  $N_f \sim N_c$ .

A second point related to the smearing is: can we consider the smearing as an approximation when we wrote above, that for example in the UV the distance between flavor branes grows unbounded? In which sense can we consider a ‘continuous distribution’ of flavor branes? Perhaps, in this sense, the smearing should be considered as the s-wave, in a multipole expansion of the true solution. Nevertheless, if we insist on the superpotential of the field theory with flavors to have the same symmetries that the unflavored theory has (see Section 5.1 of [9]) the smearing will be necessary. In the case in which the superpotential breaks the isometries of the original unflavored theory, the dual string solutions should depend-aside from the radial coordinate-on angles, hence partial differential eqs. will be describing the dynamics, making the problem more complicated.

Another point on which we would like to comment is related to the T-dual version of our set-up. As is well known (see for example [39]), the IIA version of our type of field theories is represented by an array of NS5, D6 and D4 branes. The positions of the D6 branes are arbitrary, one can have them coincident or separate them (at the only cost of breaking the  $SU(N_f)$  symmetry). In our type IIB construction we perform the smearing discussed above to find a simple solution. This seems to point to a particular distribution of six branes in the T-dual picture. Nevertheless, it may be possible to find solutions where non-uniform configurations of flavor branes are smeared (an example in a different model is in the paper [38]). The different smearings should correspond to different distributions of D6 branes in the T-dual version. The influence (or not) of the different smearings over the IR of the solution should teach us about features of the field theory and the stringy completion.

Finally an issue that is occasionally raised is the possible existence of tadpoles. This question mostly arises when the flavor branes are D7’s as it happens in some of the models discussed in [14]. One must find a (closed) cycle inside  $\Sigma_p$ -where the flavor branes are pointlike objects-to

compute the integral  $\int F_{RR}$  and this should be zero to avoid tadpoles. One can see that such manifold does not exist or that integral is zero in the case of D5 branes considered in this paper or in the models using D7's flavor branes on the conifold.

## 7.1 Conclusions

In this paper we have presented a unified way of working with different type IIB String backgrounds conjectured to be dual to a version of  $N = 1$  SQCD. We have presented various new solutions, including a new exact solution. We also studied different field theory aspects of the new solutions, providing among other new things a systematic criterion for string breaking (screening) and a new understanding of UV properties in the case in which the theory is deformed by an irrelevant operator. We also extended the treatment to the case of  $SO(N_c)$  gauge groups, we checked that many immediate properties are satisfied. Interestingly enough, we proposed a candidate object to compute the Wilson loop in the spinorial representation. This object, shows confinement as predicted on general grounds.

It should be nice to extend our studies in the following directions:

- To extend our analysis in Section 3 to the case of massive flavors and to construct duals to SQCD using type IIA backgrounds [40].
- To explore new gauge theory aspects of the new solutions presented in Section 4 and the new String-inspired objects mentioned in Section 6.
- To find extensions of the set-up presented in this paper to study different versions of SQCD, with different superpotentials, in different vacua, etc.
- To apply this line of research to possible strongly coupled field theories in Physics beyond the Standard Model.

We leave these for future work.

## 8 Acknowledgments

We benefited from discussions with various colleagues, whom we gratefully thank. A partial list includes: Adi Armoni, Ofer Aharony, Francesco Bigazzi, Aldo Cotrone, Alex Buchel, Amit Giveon, David Kutasov, Prem Kumar, Asad Naqvi, Alfonso Ramallo, Angel Paredes, Diana Vaman and Pedro Silva.

## A Appendix: Some aspects of the QFT

In this appendix we comment on some aspects of the field theory described in Section 2.1.

Let us recall that the field theory contains an  $N = 1$  massless vector multiplet  $\mathcal{W}_\alpha = (\lambda, A_\mu)$  plus a tower of massive KK chiral fields denoted by  $\Phi_k = (\phi_k, \psi_k)$  and massive vector multiplets  $W_k$ .<sup>13</sup>

It is useful to give an assignment of R-charges to the fields in a massive vector multiplet,

$$R[V^0] = R[A] = R[v_\mu] = R[D] = 0, \quad R[\lambda] = -R[\psi] = 1, \quad R[F] = -R[F^*] = -2. \quad (\text{A.2})$$

The reader should wonder how is it possible that a theory like (2.1), so different to N=1 SYM, can reproduce so many non-perturbative results of SYM (obtained using the dual description, that is the string background in [7]). Indeed, it was shown in [28], by using the string dual description of the field theory, that many of the observables are such that the contribution of the KK modes is zero.

The question can be posed like this: is there a QFT way of understanding why the KK modes do not contribute to typical quantities like the beta function, the R symmetry anomaly, etc?

For anomalies, one can take the simple view that only massless fields will contribute to them. In this case, the only massless fermion is a Majorana spinor, that is the gaugino of N=1 SYM. So, this would produce the same anomaly of  $U(1)_R \rightarrow Z_{2N_c}$ . But if we compute things at very high energies, many of the KK modes can be considered massless, so, they should indeed count towards the anomaly computation. In other words, since the anomaly coefficient is invariant under RG flows, we may wonder how it happens that all the KK fermions do not contribute.

This can be understood by analyzing the R-charge assignment of the KK fields. We assign R-charge  $R[\Phi_k] = 1$  to the massive chiral multiplets, then the fermions in the multiplets will have  $R[\psi_k] = 0$ , so, they will not contribute to any triangle involving the R-charge. For the massive vector multiplets, we have that the two fermions inside them have R-charges  $R[\lambda_k] = -R[\psi_k] = 1$ , then the multiplet cancels within itself. This may be the reason why we get the ‘correct’ R-symmetry pattern (that is the one of N=1 SYM).

Notice that with the previous R-charge assignment, it is clear that if the superpotential (like the one proposed in Section 2.1) has interactions of three superfields, or higher order interactions with the massive vectors, the R-symmetry is explicitly broken, because the coupling constants are charged. This could be thought as spontaneous symmetry breaking. But we should still have an ‘effective Lagrangian’, that is what the supergravity solution provides, that matches the anomalies.

---

<sup>13</sup>A massive vector superfield has the degrees of freedom of a vector superfield together with the ones of a chiral multiplet. That is,

$$\begin{aligned} V^0 = & \frac{1}{2}A^0 + \sqrt{2}[\theta\bar{\psi} + \bar{\theta}\psi] - \theta\sigma^\mu\bar{\theta}v_\mu + \theta^2F + \bar{\theta}^2F^* + \\ & i\theta^2\bar{\theta}[\bar{\lambda} + \frac{1}{\sqrt{2}}\sigma^\mu\partial_\mu\psi] - i\bar{\theta}^2\theta[\lambda + \frac{1}{\sqrt{2}}\sigma^\mu\partial_\mu\bar{\psi}] + \theta^2\bar{\theta}^2(D + \nabla^2A^0). \end{aligned} \quad (\text{A.1})$$

From here we can construct, as usual, the field strength  $W_k$ .

Now, let us focus on beta functions. The question here is how is it possible that the papers [26] get the correct NSVZ beta function. One possibility is that all the massive fields have in the far UV (when they could be considered massless) anomalous dimensions such that they do not contribute to the beta function. Indeed, the NSVZ result for the Wilsonian beta function,

$$\beta_g = -\left[3N_c + N_k(1 - \gamma_k)\right], \quad (\text{A.3})$$

shows that if the anomalous dimensions of each massive field behaves in the UV as

$$\gamma_k \sim 1 + O(m_k/E), \quad (\text{A.4})$$

then, the field will not contribute to the anomalous dimension.

Let us now consider the theory after the quark  $Q$  and anti-quark superfields  $\tilde{Q}$  have been added to the theory. The superpotential is then,

$$\mathcal{W} = \sum_{ijk,abc} z_{ijk}^{abc} \Phi_i^{ab} \Phi_j^{bc} \Phi_k^{ca} + \sum_k \hat{f}(\Phi_k) W_{k,\alpha} W_k^\alpha + \sum_k \mu_k |\Phi_k^{ab}|^2 + \sum_{r,ab} \kappa_{ij,(r)} \tilde{Q}_i^a \Phi_r^{ab} Q_j^b, \quad (\text{A.5})$$

and the F-term eqs. for the massive KK modes will be,

$$2\mu_p \Phi_p^{ab} + \left[ 3! \sum_{c,jk} z_{pjk}^{abc} \Phi_j^{ac} \Phi_k^{cb} + \frac{\partial \hat{f}(\Phi)}{\partial \Phi_p^{ab}} W_k W_k + \sum_{ab} \kappa_{ij,(p)} \bar{Q}^{a,i} Q^{b,j} \right] = 0, \quad (\text{A.6})$$

and for the massive vector,

$$\partial_{W_\alpha} \mathcal{W} = \hat{f} W_k = 0. \quad (\text{A.7})$$

So, we impose that there is a solution of the form

$$W_k = 0, \quad \Phi_p^{ab} = -\frac{\kappa_{(p),ij}}{2\mu_p} \tilde{Q}_i^a Q_j^b = \frac{\kappa_{p,ij}}{\mu_p} M^{ij} \delta^{a,b} = -\frac{\kappa_p}{2\mu_p} M \delta^{ab}, \quad (\text{A.8})$$

where we used that the constants  $z_{ijk}^{abc}$  are antisymmetric in both sets of indexes, and the  $W_k = 0$  in the vacuum. When replacing this into the superpotential, we find an effective superpotential that is quartic in the quarks or quadratic in the meson fields

$$W_{eff} \sim -\sum_p \frac{\kappa_{(p),ij}^2}{2\mu_p^2} (\bar{Q}_i Q_j)^2 \sim \frac{\kappa^2}{2\mu} M^2. \quad (\text{A.9})$$

In Section 2.1, we used that all the  $\kappa_p$  are equal, because we made no distinction between quarks due to the uniform smearing (likely, there is a phase in the coupling for each quark that we are neglecting here!). In the coefficient  $\mu$  we are counting the sum of all the masses that have been integrated out, with its respective degeneracy.

## B Appendix: Expansions in integration constants

### B.1 Large $c_+$ expansion

In this appendix we construct systematically the  $c_+$  expansions of both type **A** and type **N** solutions and show that the former asymptote to the conifold, while the latter to the deformed conifold. As we will see, these expansions resum certain terms in the class II UV expansions (4.15)-(4.16).

Let us start by observing that eq. (3.20) for  $P$  can be written in the form

$$\partial_\rho (s(P^2 - Q^2)(P' + N_f)) + 4s(P' + N_f)(N_f P + QQ') = 0, \quad (\text{B.1})$$

where

$$s(\rho) = \begin{cases} e^{-4\rho}, & \rho_o \rightarrow -\infty, \\ \sinh^2 \tau, & \rho_o > -\infty. \end{cases} \quad (\text{B.2})$$

Integrating (B.1) twice we obtain

$$\begin{aligned} P^3 - 3Q^2 P + 3 \int d\rho (2QQ' P + N_f(P^2 - Q^2)) \\ + 12 \int d\rho s^{-1} \int d\rho s(P' + N_f)(N_f P + QQ') = c_- + 4c_+^3 \int d\rho s^{-1}, \end{aligned} \quad (\text{B.3})$$

where  $c_\pm$  are arbitrary integration constants. Now

$$4 \int d\rho s^{-1} = \begin{cases} e^{4\rho}, & \rho_o \rightarrow -\infty, \\ \frac{1}{2} (\sinh(4(\rho - \rho_o)) - 4(\rho - \rho_o)), & \rho_o > -\infty, \end{cases} \quad (\text{B.4})$$

and so in both cases  $\int d\rho s^{-1} \sim e^{4\rho}$  as  $\rho \rightarrow \infty$ . Since  $Q$  only goes linearly with  $\rho$ , it follows from (B.3) that solving (B.3) in a large  $c_+$  expansion should precisely reproduce the class II UV expansions (4.15) and (4.16), but it should resum certain terms associated with  $s(\rho)$  in the type **N** case. Inserting an ansatz of the form

$$P = \sum_{n=0}^{\infty} c_+^{1-n} P_{1-n}, \quad (\text{B.5})$$

in (B.3) we obtain

$$P_1 = \begin{cases} e^{4\rho/3} \\ \frac{1}{2^{1/3}} (\sinh(4(\rho - \rho_o)) - 4(\rho - \rho_o))^{1/3} \end{cases} = \begin{cases} e^{4\rho/3}, & \rho_o \rightarrow -\infty, \\ \sinh(2(\rho - \rho_o)) \mathcal{K}(\rho - \rho_o), & \rho_o > -\infty, \end{cases} \quad (\text{B.6})$$

where  $\mathcal{K}(\rho)$  is the function introduced in Appendix C of [9] and corresponds to the deformed



conifold geometry. Solving recursively for the subleading terms one determines

$$\begin{aligned}
P_0 &= -N_f P_1^{-2} \left( \int d\rho P_1^2 + 4 \int d\rho s^{-1} \int d\rho s P_1 P_1' \right), \\
P_{-1} &= -\frac{1}{3} P_1^{-2} \left( 3P_1 P_0^2 + 6N_f \int d\rho P_1 P_0 + 12N_f \int d\rho s^{-1} \int d\rho s (P_1 P_0' + P_1' P_0) \right. \\
&\quad \left. - 3Q^2 P_1 + 6 \int d\rho Q Q' P_1 + 12 \int d\rho s^{-1} \int d\rho s (Q Q' P_1' + N_f^2 P_1) \right), \\
P_{-2} &= -\frac{1}{3} P_1^{-2} \left( 6P_1 P_0 P_{-1} + P_0^3 + 3N_f \int d\rho (2P_1 P_{-1} + P_0^2) \right. \\
&\quad + 12N_f \int d\rho s^{-1} \int d\rho s (P_1 P_{-1}' + P_1' P_{-1} + P_0 P_0') - 3Q^2 P_0 + 6 \int d\rho Q Q' P_0 \\
&\quad + 12 \int d\rho s^{-1} \int d\rho s (Q Q' P_0' + N_f^2 P_0) - 3N_f \int d\rho Q^2 \\
&\quad \left. + 12N_f \int d\rho s^{-1} \int d\rho s Q Q' + c_- \right), \\
P_{-n-2} &= -\frac{1}{3} P_1^{-2} \left\{ \sum_{m=1}^{n+2} \left( 2P_1 P_{1-m} P_{m-n-2} + \sum_{k=1}^{n-m+3} P_{1-m} P_{1-k} P_{m+k-n-2} \right) \right. \\
&\quad + 3N_f \sum_{m=0}^{n+2} \left( \int d\rho P_{1-m} P_{m-n-1} + 4 \int d\rho s^{-1} \int d\rho s P_{1-m} P_{m-n-1}' \right) - 3Q^2 P_{-n} \\
&\quad \left. + 6 \int d\rho Q Q' P_{-n} + 12 \int d\rho s^{-1} \int d\rho s (Q Q' P_{-n}' + N_f^2 P_{-n}) \right\}, \quad n \geq 1. \quad (\text{B.7})
\end{aligned}$$

For the type **A** case these recursion relations reproduce the asymptotic expansion (4.15) and so for these backgrounds the large  $c_+$  expansion coincides with the class II asymptotic expansion. For the type **N** backgrounds, however, the  $c_+$  expansion resums certain terms in the asymptotic expansion (4.16). In particular, the expansion in the parenthesis multiplying  $c_+$  in (4.16) is now identified with the asymptotic expansion of  $\frac{1}{2^{1/3}} (\sinh(4(\rho - \rho_o)) - 4(\rho - \rho_o))^{1/3}$ , after setting  $\rho_o = 0$  and absorbing an overall constant in  $c_+$ .

## B.2 Small $c_+$ expansion

In this appendix we systematically construct the most general solution of (3.20) in the vicinity of the singular solution (4.19), in a small  $c_+$  expansion. We start by inserting an expansion of the form

$$P = \sum_{n=0}^{\infty} c_+^{3n} P_n, \quad (\text{B.8})$$

where  $P_0$  is given by (4.19), in (B.1). Solving the resulting equations recursively we find that, provided

$$\begin{aligned}\Omega &\equiv P_0^2 - Q^2 \\ &= \left( (P_o - Q_o \cosh \tau) - \frac{2N_c - N_f}{2} (\cosh \tau - 1) - (N_f + (2N_c - N_f) \cosh \tau) \rho \right) \times \\ &\quad \left( (P_o + Q_o \cosh \tau) + \frac{2N_c - N_f}{2} (\cosh \tau - 1) - (N_f - (2N_c - N_f) \cosh \tau) \rho \right), \quad (\text{B.9})\end{aligned}$$

does not vanish identically, then

$$\begin{aligned}P_1 &= \int d\rho s^{-1} \Omega, \\ P_n &= \int d\rho s^{-1} \Omega \int d\rho \Omega^{-2} R_n, \quad n \geq 2,\end{aligned} \quad (\text{B.10})$$

where

$$\begin{aligned}R_n &= -\frac{1}{3} \sum_{m=1}^{n-1} \left\{ \partial_\rho \left[ s \left( \partial_\rho \left( 2P_0 P_m P_{n-m} + \sum_{k=1}^{n-m} P_m P_k P_{n-m-k} \right) + 3N_f P_m P_{n-m} \right) \right] \right. \\ &\quad \left. + 12N_f s P_m P'_{n-m} \right\}, \quad n \geq 2.\end{aligned} \quad (\text{B.11})$$

Evaluating the first few terms for the type **A** case ( $\rho_o \rightarrow \infty$ ) we find

$$\begin{aligned}P_0 &= -N_f \rho + P_o, \\ P_1 &= e^{4\rho} \left( N_c(N_f - N_c) \rho^2 - \frac{1}{2} (N_c(N_f - N_c) + N_f P_o + (2N_c - N_f) Q_o) \rho \right. \\ &\quad \left. + \frac{1}{4} (P_o^2 - Q_o^2) + \frac{1}{8} (N_f P_o + (2N_c - N_f) Q_o + N_c(N_f - N_c)) \right), \\ P_2 &= \dots,\end{aligned} \quad (\text{B.12})$$

which coincides with the type I IR expansion of [12] (cf. eqs. (3.7)-(3.9)). In this case the IR is located at  $\rho \rightarrow -\infty$ .

For the type **N** case, we note that since  $P_0$  contains the arbitrary constant  $P_o$ , we can always extend this solution in the IR up to the smallest possible value of the radial coordinate, i.e.  $\rho_o$ , by a suitable shift in the constant  $P_o$ . However,  $P$  remains finite at  $\rho_o$ , while from (3.17) we see that  $Q$  has a pole at  $\rho_o$ , unless  $Q_o + (2N_c - N_f)/2 = 0$ . In order to get a well defined solution then, where both  $H = (P + Q)/4$  and  $G = (P - Q)/4$  remain positive, we must ensure that  $Q$  has no pole at  $\rho_o$  and so we must set  $Q_o = -(2N_c - N_f)/2$ . Taking, without

loss of generality, then  $\rho_o = 0$ , we obtain

$$\begin{aligned}
P_0 &= -N_f \rho + P_o, \\
P_1 &= \frac{1}{48} \left\{ 8 \left( (2N_c - N_f)^2 + N_f^2 \right) \rho^3 + 24N_f P_o \rho^2 + 6 \left( (2N_c - N_f)^2 - 4P_o^2 \right) \rho - 3N_f P_o \right. \\
&\quad \left. - \frac{3}{2} \left[ 16N_c(N_c - N_f)\rho^2 + 8N_f P_o \rho + 5(2N_c - N_f)^2 - 3N_f^2 - 4P_o^2 \right] \sinh(4\rho) \right. \\
&\quad \left. + 3 \left[ (3(2N_c - N_f)^2 - N_f^2) \rho + N_f P_o \right] \cosh(4\rho) \right\}, \\
P_2 &= \dots
\end{aligned} \tag{B.13}$$

where we have chosen the integration constant in  $P_1$  such that  $P_1 \rightarrow 0$  as  $\rho \rightarrow 0$ . Expanding this around  $\rho = 0$  then one obtains

$$\begin{aligned}
P_0 &= -N_f \rho + P_o, \\
P_1 &= \frac{4}{3} P_o^2 \rho^3 - 2N_f P_o \rho^4 + \frac{4}{5} \left( \frac{4}{3} P_o^2 + N_f^2 \right) \rho^5 + \mathcal{O}(\rho^6), \\
P_2 &= \dots,
\end{aligned} \tag{B.14}$$

Identifying then  $2c_+ P_o^2 = c_2 N_c$  and  $c_1 = 4(2N_c - N_f)/3P_o$  the expansion (B.14) exactly reproduces the expansion (4.21) in [9]. The expansion (B.13), however, resumming certain terms of this IR expansion, extends the solution to a wider range of the radial coordinate.

Finally, note that in the type **A** case ( $\rho_o \rightarrow -\infty$ ) it is possible that  $\Omega = 0$  identically. This happens when either  $N_f = N_c$  and  $P_o = -Q_o$  or  $N_c = 0$  and  $P_o = Q_o$ . The solution is then obtained recursively by setting  $R_n = 0$ ,  $n \geq 2$ , which gives

$$\begin{aligned}
P_0 &= -N_f \rho + P_o, \\
P_1 &= e^{2\rho} \sqrt{-N_f \rho + P_o + N_f/4}, \\
P_2 &= -\frac{1}{2P_1} \int d\rho e^{4\rho} P_0 \int d\rho P_0^{-2} \partial_\rho (e^{-4\rho} P_1 \partial_\rho P_1^2), \\
P_3 &= \dots
\end{aligned} \tag{B.15}$$

$P_2$  can be expressed in terms of the error function, but the integrals involved in  $P_3$  and higher orders cannot be done in closed form in general. This solution is valid for  $\rho > -\infty$  and it is a new type of IR behavior for the type **A** backgrounds.

## References

- [1] J. M. Maldacena, Adv. Theor. Math. Phys. **2**, 231 (1998) [Int. J. Theor. Phys. **38**, 1113 (1999)]; hep-th/9711200.

- [2] S. S. Gubser, I. R. Klebanov and A. M. Polyakov, Phys. Lett. B **428**, 105 (1998); hep-th/9802109.
- [3] E. Witten, Adv. Theor. Math. Phys. **2**, 253 (1998); hep-th/9802150.
- [4] N. Itzhaki, J. M. Maldacena, J. Sonnenschein and S. Yankielowicz, Phys. Rev. D **58**, 046004 (1998); hep-th/9802042. H. J. Boonstra, K. Skenderis and P. K. Townsend, JHEP **9901**, 003 (1999) [arXiv:hep-th/9807137].
- [5] E. Witten, Adv. Theor. Math. Phys. **2**, 505 (1998) [arXiv:hep-th/9803131].
- [6] J. P. Gauntlett, N. Kim, D. Martelli and D. Waldram, Phys. Rev. D **64**, 106008 (2001) [arXiv:hep-th/0106117]. F. Bigazzi, A. L. Cotrone and A. Zaffaroni, Phys. Lett. B **519**, 269 (2001); hep-th/0106160.
- [7] J. M. Maldacena and C. Nunez, Phys. Rev. Lett. **86**, 588 (2001); hep-th/0008001.
- [8] A. H. Chamseddine and M. S. Volkov, Phys. Rev. Lett. **79**, 3343 (1997); hep-th/9707176.
- [9] R. Casero, C. Nunez and A. Paredes, Phys. Rev. D **73**, 086005 (2006) [arXiv:hep-th/0602027].
- [10] See A. Karch and E. Katz, JHEP **0206**, 043 (2002) [arXiv:hep-th/0205236]. For updated reviews see J. Erdmenger, N. Evans, I. Kirsch and E. Threlfall, Eur. Phys. J. A **35**, 81 (2008) [arXiv:0711.4467 [hep-th]] and D. Rodriguez-Gomez, Int. J. Mod. Phys. A **22**, 4717 (2007) [arXiv:0710.4471 [hep-th]].
- [11] C. Nunez, A. Paredes and A. V. Ramallo, JHEP **0312**, 024 (2003); hep-th/0311201.
- [12] R. Casero, C. Nunez and A. Paredes, Phys. Rev. D **77**, 046003 (2008) [arXiv:0709.3421 [hep-th]].
- [13] R. P. Andrews and N. Dorey, Nucl. Phys. B **751**, 304 (2006) [arXiv:hep-th/0601098]. R. P. Andrews and N. Dorey, Phys. Lett. B **631**, 74 (2005) [arXiv:hep-th/0505107].
- [14] A. Paredes, JHEP **0612**, 032 (2006) [arXiv:hep-th/0610270]. F. Benini, F. Canoura, S. Cremonesi, C. Nunez and A. V. Ramallo, JHEP **0702**, 090 (2007) [arXiv:hep-th/0612118]. G. Bertoldi, F. Bigazzi, A. L. Cotrone and J. D. Edelstein, Phys. Rev. D **76**, 065007 (2007) [arXiv:hep-th/0702225]. S. Hirano, JHEP **0705**, 064 (2007) [arXiv:hep-th/0703272]. F. Benini, F. Canoura, S. Cremonesi, C. Nunez and A. V. Ramallo, JHEP **0709**, 109 (2007) [arXiv:0706.1238 [hep-th]]. A. L. Cotrone, J. M. Pons and P. Talavera, JHEP **0711**, 034 (2007) [arXiv:0706.2766 [hep-th]]. B. A. Burrington, V. S. Kaplunovsky and J. Sonnenschein, JHEP **0802**, 001 (2008) [arXiv:0708.1234 [hep-th]]. D. f. Zeng, arXiv:0708.3814 [hep-th]. F. Benini, arXiv:0710.0374 [hep-th]. O. Lorente-Espin and

- P. Talavera, JHEP **0804**, 080 (2008) [arXiv:0710.3833 [hep-th]]. F. Canoura, P. Merlatti and A. V. Ramallo, JHEP **0805**, 011 (2008) [arXiv:0803.1475 [hep-th]]. P. Basu and A. Mukherjee, arXiv:0803.1880 [hep-th]. S. Cremonesi, arXiv:0805.4384 [hep-th].
- [15] P. Koerber and D. Tsimpis, JHEP **0708**, 082 (2007) [arXiv:0706.1244 [hep-th]].
  - [16] E. Caceres, R. Flauger, M. Ihl and T. Wrase, JHEP **0803**, 020 (2008) [arXiv:0711.4878 [hep-th]].
  - [17] N. Seiberg, Nucl. Phys. B **435**, 129 (1995) [arXiv:hep-th/9411149].
  - [18] M. J. Strassler, arXiv:hep-th/0505153.
  - [19] K. A. Intriligator, Nucl. Phys. B **580** (2000) 99 [arXiv:hep-th/9909082].
  - [20] K. Skenderis and M. Taylor, JHEP **0608** (2006) 001 [arXiv:hep-th/0604169].
  - [21] J. M. Maldacena, Phys. Rev. Lett. **80**, 4859 (1998) [arXiv:hep-th/9803002]. S. J. Rey and J. T. Yee, Eur. Phys. J. C **22**, 379 (2001) [arXiv:hep-th/9803001].
  - [22] J. Sonnenschein, arXiv:hep-th/0003032.
  - [23] R. Apreda, F. Bigazzi, A. L. Cotrone, M. Petrini and A. Zaffaroni, Phys. Lett. B **536**, 161 (2002) [arXiv:hep-th/0112236].
  - [24] A. Armoni, arXiv:0805.1339 [hep-th].
  - [25] F. Bigazzi, A. L. Cotrone, C. Nunez and A. Paredes, arXiv:0806.1741 [hep-th].
  - [26] P. Di Vecchia, A. Lerda and P. Merlatti, Nucl. Phys. B **646**, 43 (2002) [arXiv:hep-th/0205204]. M. Bertolini and P. Merlatti, Phys. Lett. B **556**, 80 (2003) [arXiv:hep-th/0211142].
  - [27] N. Seiberg, Phys. Lett. B **390** (1997) 169 [arXiv:hep-th/9609161]. U. H. Danielsson, G. Ferretti, J. Kalkkinen and P. Stjernberg, Phys. Lett. B **405**, 265 (1997) [arXiv:hep-th/9703098]. D. I. Kazakov, JHEP **0303**, 020 (2003) [arXiv:hep-th/0209100].
  - [28] U. Gursoy and C. Nunez, Nucl. Phys. B **725**, 45 (2005) [arXiv:hep-th/0505100].
  - [29] J. Gomis, Nucl. Phys. B **624**, 181 (2002) [arXiv:hep-th/0111060].
  - [30] A. Armoni, D. Israel, G. Moraitis and V. Niarchos, Phys. Rev. D **77**, 105009 (2008) [arXiv:0801.0762 [hep-th]].
  - [31] K. A. Intriligator and N. Seiberg, Nucl. Phys. B **444**, 125 (1995) [arXiv:hep-th/9503179].

- [32] G. Carlino, K. Konishi, S. P. Kumar and H. Murayama, Nucl. Phys. B **608**, 51 (2001) [arXiv:hep-th/0104064].
- [33] E. Witten, JHEP **9807** (1998) 006 [arXiv:hep-th/9805112].
- [34] A. Hanany and B. Kol, JHEP **0006** (2000) 013 [arXiv:hep-th/0003025].
- [35] A. Capella, U. Sukhatme, C. I. Tan and J. Tran Thanh Van, Phys. Rept. **236**, 225 (1994).
- [36] G. 't Hooft, Nucl. Phys. B **72**, 461 (1974).
- [37] G. Veneziano, “Some Aspects Of A Unified Approach To Gauge, Dual And Gribov Theories,” Nucl. Phys. B **117**, 519 (1976).
- [38] F. Bigazzi, A. L. Cotrone and A. Paredes, arXiv:0807.0298 [hep-th].
- [39] A. Giveon and D. Kutasov, Rev. Mod. Phys. **71**, 983 (1999) [arXiv:hep-th/9802067].
- [40] M. Atiyah, J. M. Maldacena and C. Vafa, J. Math. Phys. **42**, 3209 (2001) [arXiv:hep-th/0011256]. J. D. Edelstein and C. Nunez, JHEP **0104**, 028 (2001) [arXiv:hep-th/0103167].